ВОПРОСЫ ТЕОРИИ

Stochastic Seigniorage and Sustainable Debt in the Economy of Transition

Smirnov A.D.

Stochastic seigniorage, government debt and borrowing dynamics are modeled in the context of a transition economy. Rational investors assessing market risks, hedge against them and construct riskless portfolios of seigniorage and debts. Risk-neutral asset valuation made it possible to formulate equilibrium conditions for the debt sustainability. Lending to the government is considered as a perpetual American call option that is optimized with respect to seigniorage without defaulting on the debt outstanding. The model is applied to the August 1998 domestic debt default in Russia to determine short-run effects of seigniorage upon the government debt sustainability.

In the last decade Russian government had a dramatic experience with domestic debt and borrowing on the open market [19]. The market for domestic debt had been virtually nonexistent until 1994 when the government started to borrow heavily by issuing securities with short maturity resembling three-month T-bills. Borrowing helped to cap inflation in the medium run, but exploding growth of the debt made it *de facto* useless for the budget deficit financing. Since the late 1997 borrowing was used almost exclusively for rolling the debt over until it became unsustainable, and the government was forced in August 1998 to declare a formal default on its domestic debt obligations.

Yet domestic debt market revived and total amount of government borrowing in 2002 became almost equal to the amount of borrowing in 1998 though with better qualitative characteristics (debt-to-GDP ratio, composition of debts, its average maturity, YTMs, etc). Debt management to be an efficient part of macroeconomic performance calls for, in our opinion, much better coordination of fiscal and monetary policy. Hence the modeling of government debt and borrowing continues to be an important part of the study of the transition of Russian economy towards market.

The Russian default clearly outlined several questions concerning risks on the market of government debts, their assessment by private investors in the process of lending to the government, debt sustainability and guaranties of its redemption, reliability of a government as a borrower. These and more questions arise in the short run, and long run consequences notwithstanding, the plausible answers to them has the nontradability of the total government debt outstanding (macrodebt, in short), for

Smirnov A.D. – Professor of the State University – Higher School of Economics (Moscow) and member of Russian Academy of Natural Sciences.

the latter cannot be sold or bought as a whole. But analogies with the debt instruments yet to remain sound, especially if relations were established properly between debt, borrowing and money in the short run on the macrolevel. The idea of the «real options» theory seems to be most helpful in this respect [6].

Review of Methodology

The literature on the debt analysis, either sovereign or corporate, is vast and incorporates many different approaches [10, 24]. The long run aspects of the debt management were thoroughly studied in macroeconomic literature, including [16].

This model attempts to apply methodology of the diffusion analysis to the study of government debt and borrowing processes on the macrolevel in the short run. Since government debt dynamics is macroeconomic as well as financial process, it could be described in terms relevant to these aspects. Bonds and money are different entities on the micro-level but their distinction is rather blurred on the macrolevel because in effect seigniorage flow in nominal terms constitutes coupon payments on government debt[1)].

The analysis of the budget deficit financing process under uncertainty lies in the underpinning of the model. Domestic government debt is considered as a homogeneous coupon macrobond, either perpetual or of some fixed maturity; it was represented as a function of coupon payments in accordance with the standard theory of bonds [11]. Analogies with the preferred stock are also consistent. Coupon payments were, under rather weak constraints, modeled as seigniorage issuance governed by a stochastic differential equation, interpretation of which follows the Ricardian tradition. Since the «Ricardian Equivalence» does not play the pivotal role in the short run, the analysis of trade-offs among debts, taxes and seigniorage was focused on the latter as the major factor affecting market value of debts and market risks[2)]. The so called which seems to be more content with the reality of economic transition.

Budget deficit financing is analyzed as the debt service and as continuous adjustments to portfolio of assets held by private investors. In the former, the expected return on a government debt equals to the sum of coupon payments and debt appreciation (depreciation). In the latter, budget deficit represents as a portfolio of seigniorage and the opportunity to continue lending (or government borrowing). These representations are closely connected but not identical. Stochastic borrowing differs drastically from its deterministic analogue: in the deterministic model amount of «new» debts is identical to debt appreciation (depreciation) while they are different quantities for random processes. Hence in a stochastic environment they should be modeled differently.

Private investors' behavior in the economy of transition is assumed to be rational. In the short run investors are able not only to form expectations, using all the information available, but also decompose the risk-adjusted rate of return into a riskless component and risk premium. The latter implies the existence of the market price of risk in the economy of transition. The above said is equivalent, in its turn, to the ability of private investors to construct replicating portfolios and perform (per-

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[1)] We did not distinguish the government from the central bank assuming the low degree of the Russian bank independence.

[2)] Fiscal aspects are studied in [5, 25].

fect) hedging to beat the adverse effects of market risks. The implementation of riskless replicating portfolios, including risk-neutral asset valuation, is viewed as the dominant strategy used by investors on the financial markets. Quite evidently, the validity of this hypothesis is disputable, being conditional to the knowledge about structure and performance of emerging markets.

With regard to the sustainability of government debt in the short run we follow the methodology proposed by F. Black and M. Sholes [1] and R. Merton [13] for the financial distress analysis. An attempt was made to adapt their approach to the study of macroeconomic phenomenon of a government domestic debt dynamics. Since holding debt is risky, it could be decomposed into debt outstanding and the value of the opportunity to continue lending (or government borrowing); the latter being interpreted as a call option. Short-run equilibrium is represented as equality between debt outstanding and the opportunity to make new loans, on the costs side, with the market value of debt and the contingent debt guaranty, as returns to investors. The debt guaranty is equivalent to the value of the put-to-default option One of the major problems that arise in this respect is the validity of no-arbitrage approach with respect to the emerging financial market of Russian economy in transition. The equilibrium equation forms the condition for the debt sustainability, if private investors and a government follow a coherent strategy.

Were the debt and government borrowings depend upon seigniorage issuance, the latter becomes an instrument of debt sustainability within some range. Assuming perpetual debt and perpetual American option to continue lending, the amplification of seigniorage makes a feasible strategy for a government as a reliable borrower. The sustainable debt can be represented as the to the optimal stopping problem. As the appropriate Dixit–Pindyke [7] methodology suggests, analysis of debt sustainability might be performed either via a dynamic programming or a contingent claims technique, because equation that governs the dynamics of the perfectly hedged portfolio appears to be the Bellman equation. Increasing inflationary pressure is inevitable in the case of additional seigniorage issuance, and the trade-offs between inflation and domestic government debt accumulation are to be considered within some politically determined preferences [14].

In macroeconomic aspect the model attracted attention to the short run effects of monetary policy upon debt holders. The most important fact seems to be that in economic transition «tight money» policy might force even *a priory* reliable borrower to renege on its obligations. Conversely, by exploiting rational motivations of debt holders the government might redeem its debt in full. The model suggested a strategy of default avoidance in the short run using monetary instruments: roughly speaking, by making money easier the government increases probability of the domestic debt redemption. The strategy of hedging performed by the debt holders decreases to some extent the inflationary pressure that appears as a result of seigniorage issuance.

The paper contains also some empirical estimations of the Russian government domestic debt performance in order to validate the model and to provide some conclusions about the August 1998 default. Russian default was studied in many monographs and papers, for example, in [8, 10]. Many Russian and Western economists thoroughly studied one consequence of a «tight money» policy: the emergence of huge and persistent arrears, and the deep demonetization of Russian economy in 1996–1998. Yet another consequence of the same policy – the inability of Russian government to pay out with roubles its debt denominated in domestic currency – left virtually unnoticed.

Stochastic debt and seigniorage

The uncertainty of economic transition is captured in the model by seigniorage representation as a continuous geometric random process (*GRP*), s_t, being governed by the stochastic differential equation:

$$\frac{ds_t}{s_t} = a\,dt + \sigma\,dW_t. \qquad (1)$$

In Eq. (1) $a = \mathrm{E}_t^P[\frac{ds_t}{s_t}]$ is the expected rate of seigniorage issuance under the «true» probability measure P; σ is the parameter of seigniorage volatility; and W_t is a standard Wiener process that captures all the random shocks that influences seigniorage issuance. Since the study is focused on the debt problems, all the processes are in nominal terms[3)].

By applying Ito's lemma and the formula for the lognormal distribution the expected value of seigniorage becomes

$$\mathrm{E}^P[s_t] = s_0 \exp\{at\}. \qquad (2)$$

Assuming seigniorage issuance to be an infinite process, the present value of the future stream of seigniorage, $b(s_t, t) = b(s_t)$, has the following representation:

$$b(s_t, t) = \mathrm{E}_t^P\{\int_t^{\infty} s(z)\exp[-\mu(z-t)]dt\} \qquad (3)$$

where $\mu > 0$ is the rate of discount. Were the government made debts, then the present value of a never matured macrobond with coupon payments, s_t, is given by Eq. (3). Debt service, in its turn, over the period dt is modeled by the government budget deficit financing equation:

$$\mu\, b(s_t)dt = s_t dt + \mathrm{E}_t^P[db] \qquad (4)$$

where application of the expectations operator E is in need since future changes in the value are unobserved as of time t. Nonrandom function of random seigniorage, $b(s_t)$, is the solution to Eq. (4), and can be interpreted as the market value of government debt. Parameter μ represents the expected rate of the debt service[4)].

[3)] The hypothesis of *GRP,* which is quite standard in financial economics [3], is not very popular among macroeconomists, though many attempts had been made in this direction (systematic bibliography might be found in [23, 7].

[4)] Eq. (4) is a modification of a standard macroeconomic model of the budget deficit financing in nominal terms:

$$\mu\ B_t = \dot{B}_t + \dot{M}_t\ ;\ \dot{M}_t \equiv s_t\ ;\ G_t - T_t = 0\,,$$

where B is risky debt outstanding; s_t is seigniorage.

The same parameter, μ, is associated with the expected risk-adjusted rate of return on a risky asset $b(s_t)$; it was used as the discount rate in (3). Since the expected rate of seigniorage growth is equal to a, the difference:

(5) $$\delta = \mu - a > 0$$

has economic meaning of coupon payments on asset $b(s_t)$, or the convenience yield in the Dixit–Pindyck terminology. In any case parameter $\mu > 0$ cannot be considered as «the return on money» which would have made Eq. (3) senseless, if money does not pay any convenience yield.

According to Eq. (4) during the infinitesimally short time period earnings of private investors (creditors to the government) consist of coupon payments (seigniorage) and the expected changes in the market value of government debt. It differs from the standard form in one respect only: the primary budget deficit is assumed to be zero implying seigniorage issuance exclusively for the debt servicing. This is a simplification, but it permits to work with the monetary risks only: otherwise actual seigniorage and associated risks should be decomposed[5].

In the model private investors are assumed to behave rationally, that is, they are capable to decompose the risk-adjusted rate of return, μ, into the riskless component, r, and risk premium, $\lambda\sigma$:

(6) $$\mu = r + \lambda \sigma_b,$$

where λ is the market price of risk, and σ_b is the volatility on the debt market. Thus the financial market in the economy of transition is at least of the weak-form efficiency, and the market price of risk exists and can be properly assessed by private investors. Operationally that means that investors are able to construct portfolios of different assets and hedge them against the adverse risk effects.

As equilibrium relation between the debt service requirements and the sources of its financing Eq. (4) is satisfied for any volumes of debt including the steady state debt. In the case of the continuous and indefinite debt rollover its solution, $b(s,t) = b(s_t)$, is given by the linear function of a debt (never matured coupon macrobond) and its coupon payments:

(7) $$b(s_t) = \frac{1}{\delta} s_t.$$

Dependence of the debt value upon seigniorage does not imply that monetary policy is determined by the debt service considerations exclusively. In a more sophisticated versions of the model money should be decomposed into component, which is determined by the real sector development as well as component under the impact of the debt service process in the short run. Yet, the opposite is quite true: the policy of tight money might under some circumstances provoke the default on the domestic debt. In the model Eq. (7) forms the boundary condition for the process of random government borrowing to be discussed later.

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[5] Eq. (1) has to be added with the SDE for the primary budget deficit, thus making the system two-dimensional.

Replication of the riskless debt dynamics

Eq. (4) is the equilibrium relation of a sustainable debt in the short run, and it could be studied in two aspects: as a debt service relation, and as a portfolio of risky assets held by private investors. The consistency of the debt service process should be supported by the behaviour of private investors: on a financial market of transition economy a government has to rely upon their willingness to supply it with loanable funds. The latter depends upon investors' desire to hold portfolio that consists of money (seigniorage) and debts[6]. The strategy implemented by private investors determines the budget financing and debt accumulation dynamics.

Consider the SDE for the market value of government debt in the period $(t, t+dt)$. Since debt holders receive seigniorage as coupon payments, over a small time interval dt the debt value is changed as

(8) $$db_t = [\mu b(s_t) - s_t]dt + \sigma_b b(s_t)dW_t,$$

where σ_b is the risk parameter on the bond market. By application of Ito's lemma to db, substitution of (1) into it, and comparing with (8), we evaluate the coefficients on the terms dt and dW_t, respectively, as:

(9) $$\mu\, b(s_t) = s_t + a s_t b'(s_t) + \frac{1}{2}(\sigma\, s_t)^2 b''(s_t) \quad (a)$$

and $$\sigma_b b(s_t) = \sigma s_t b'(s_t)\ , \quad (b)$$

which, taking into account Eq. (6), can be transformed into

(10) $$rb(s_t) = s_t + (a - \lambda\sigma)s_t b'(s_t) + \frac{1}{2}(\sigma\, s_t)^2 b''(s_t).$$

If there exists a risk-neutral probability distribution Q, then Eq. (10) holds for the risk neutral interest rate, $r > 0$. The existence of the risk-neutral probability distribution Q plays a crucial role in evaluation of debt and borrowing processes, though its existence in the economy of transition constitute a topic that is left beyond the scope of the present paper. Being taken as granted, it makes possible upon substitutions:

(11) $$\mu \to r \quad \text{and} \quad a - \lambda\sigma = r - \delta$$

to get the following ODE for the risk-neutral debt dynamics with stochastic seigniorage:

(12) $$\frac{1}{2}\sigma^2 s_t^2 b''(s_t) + (r-\delta)s_t b'(s_t) - rb(s_t) + s_t = 0\cdot$$

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[6] The same result can be achieved through analysis of a portfolio $\Psi_t = \theta_b s_t + b(s_t)$ that consists of seigniorage and debts, respectively. Parameter θ_b is the hedging ratio.

Note, that $\delta = \mu - a$ and not $r - a$, so the risk-adjusted expected return on a portfolio debt-seigniorage, Ψ_t, has to be estimated. Eq. (12) is a particular case of nonhomogeneous the *Black-Sholes equation* that has a general solution:

$$b(s_t) = A_1 s_t^{\beta_1} + A_2 s_t^{\beta_2} + \frac{1}{\delta} s_t, \tag{13}$$

where $\beta_1 < 0$, $\beta_2 > 1$ are real roots of characteristic equation:

$$\frac{1}{2}\sigma^2 \beta(\beta - 1) + (r - \delta)\beta - r = 0, \tag{14}$$

and A_1, A_2 are arbitrary constants that belong to the respective roots. Relations (11) are crucial in understanding behaviour of private investors and the structure of the debt equation (12). They will be used further in the modeling of borrowing.

In order to simplify solution (13), we are going to restrict ourselves with the steady state debt only; thus a «financial bubble» given by the complementary function in Eq. (13) has «to disappear». It can be done due to the following economic considerations. Constant A_1 was taken as zero due to the absorption condition evident from Eq. (3): debt has no market value without seigniorage, $b(0) = 0$. The second constant, A_2, is zero due to the assumption of a government non-intervention on the secondary market: it does not sell, nor buy its own debts[7]. Taking these simplifications into account, we end up with the *fundamental value* of the government debt which is the same as in Eq. (5); but this time it appeared as a result of continuous adjustments of riskless portfolios held by private investors capable of assessing risks and hedging against them. Thus private investors in a transition economy perceiving government debt as risky asset are content with its indefinite rollover. It is equivalent to the existence of a Ricardian relation (7) between seigniorage and steady state debt, which serves as a boundary condition for the process of borrowing.

Stochastic borrowing process

There is an important difference in the interpretation of equations (4) and (8). The first makes an accent on the time changes in the value of debt and implies that debt appreciation is attributed to the borrowing entirely. Its solution is equivalent to Eq. (9a), and does not take into account the effects of random factors given by Eq. (9b). Contrary to that, Eq. (8) implies that borrowing is a separate action on behalf of a government and the component db_t is attributed to debt appreciation (or depreciation) due to the effects of seigniorage issuance and random factors.

The validity of Eq. (4) implies that market participants absorb unconditionally seigniorage and new debts issued by the government (recall that the primary deficit is zero in the model). Given new debts, private investors accept any increase in domestic liquidity, which is true under the normal economic circumstances, but not always: hyperinflation is one of the counter-examples, and the «dollarisation» is another one. The occurrence of a liquidity crisis might be conceived as a cogent mani-

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[7] There were some rumours that Russian government had probed into several opportunities to buy out its debt when it floated at deep discount on the eve of the August 1998 crisis [17].

festation of the investors' reluctance to accept domestic currency, too. Though we are not concerned with these issues directly, the validity of an assumption of unconditional money acceptance seems to be too stringent for the emerging market economies, and Russia's is one of them. Hence the solution to Eq. (4) serves as a boundary condition rather then a general model of the budget deficit financing process. Government borrowing on the open market and the consequent debt accumulation are to be considered as separate, though closely interconnected processes.

In a period of transition investors lend for a short period and protect themselves from losses, if seigniorage decreases in value. The financial market, like its real life prototype in a transition economy, consists of money market and government debts of short maturity, with virtually insignificant capital market segment. Due to uncertainty of transition, private investors consider lending as risky actions to be performed and contingent upon some outcomes. Thus to model the process of borrowing (or lending, looking from the side of private investors) requires an introduction of a separate function $f(s_t, t) = f(s_t)$, which, strictly speaking, represents the value of the opportunity to lend (or to borrow).

Let us return to the basic budget deficit financing process again. This time we are going to model it as a no-arbitrage relation between the budget deficit at time t, Φ_t, seigniorage, s_t, and new government debts, $f(s_t)$, that private investors are going to acquire. A simple portfolio equation was chosen for that role:

$$\Phi_t = \theta_1 s_t + \theta_2 f(s_t), \tag{15}$$

where θ_1, θ_2 are physical quantities of seigniorage and new debts. Eq. (15) corresponds to the portfolio to be held by private investors on the financial market, and hence it reflects their responses to the policy of the budget deficit financing. One of their reactions in a transition economy is to protect themselves from losses, if seigniorage did not grow too fast. It was captured in the model through procedure of *continuous hedging*. An investor who is long in «new» debts would be short in seigniorage to hedge against losses. Losses and gains would completely offset each other, if an investor sold $f'(s_t)$ units of the underlying currency, making the value of a hedged portfolio completely predictable[8]. We are going to estimate the value of the opportunity of rational lending to the government, $f(s_t)$.

In a small period of time the value of a portfolio (15) changes as

$$d\Phi_t = \theta_1 ds_t + \theta_2 df_t. \tag{16}$$

Since seigniorage represents a random process subject to (1), the stochastic increment in the portfolio value is calculated by applying Ito's lemma and condition $(dW_t)^2 = dt$ (holding in the mean square sense):

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[8] When investors, being alarmed by some unfavourable information, start to withdrew their domestically denominated assets and convert them into foreign exchange, their spontaneous hedging works as a compensatory mechanism coherent with their reluctance to accept domestic currency. Recall that Russian default was accompanied by the domestic liquidity crisis that had two stages. The first, in November 1997, was overcome more or less successfully though reserves of Russian central bank declined dramatically, while the second ended with a sharp devaluation of the rouble in August 1998.

(17) $$df_t = f'(s_t)ds_t + \frac{1}{2}f''(s_t)(ds_t)^2 = f'(s_t)ds_t + \frac{1}{2}\sigma^2 s_t^2 f''(s_t)dt .$$

Substituting (17) into (16) for

(18) $$\theta_1 = -f'(s_t),\ \theta_2 = 1,$$

we get

(19) $$d\Phi_t = [\theta_1 + f'(s_t)]ds_t + \frac{1}{2}\sigma^2 s_t^2 f''(s_t)dt = \frac{1}{2}\sigma^2 s_t^2 f''(s_t)dt .$$

The portfolio appreciation in (19) is riskless, and, from no-arbitrage considerations, investors' expected return on a portfolio (15) has to be equal to its expected appreciation plus expected compensation for seigniorage holding at the rate $\delta > 0$:

(20) $$r\Phi_t dt = d\Phi_t + \delta\,\theta_1 s_t dt .$$

Upon rearrangement, Eq. (20) leads to the no-arbitrage equation for the riskless return on the holding of «new» debts (government borrowing):

(21) $$rf(s_t)dt = d\Phi_t + (r-\delta)s_t f'(s_t)dt .$$

It has to be repeated that as in the case of debts, the complete predictability of Eq. (21) is due to an assumption of existence of the riskless probability distribution Q under which it takes place. Eq. (21) can be expressed as the second order differential equation with respect to function $f(s_t)$:

(22) $$\frac{1}{2}\sigma^2 s_t^2 f''(s_t) + (r-\delta)s_t f(s_t) - rf(s_t) = 0 ,$$

which is actually the *Black–Sholes equation* for the particular case of perpetual government borrowing. It summarizes the complex process of mutual adjustments of a government and private investors in the process of the budget deficit financing[9]. The first two components of (22) represent the investors' risk neutral expectations of government borrowing appreciation: that is what private investors get due to the seigniorage issuance at the rate, $a > 0$.

Value of borrowing (lending), $f(s_t)$, forms a solution to homogeneous equation (22), which can be expressed as

(23) $$f(s_t) = B_1 s_t^{\beta_1} + B_2 s_t^{\beta_2} ,$$

where $\beta_1 < 0$, $\beta_2 > 1$ are characteristic roots of the same equation (14), and B_1, B_2 are arbitrary constants that belong to the respective roots. Constant B_1 is taken as zero from the absorption considerations: if no coupon payments are expected, the possibility to borrow might be considered to be equal to zero. Hence the value of borrowing (lending) is reduced to a function:

[9] It can be noted that the same result can be obtained directly from equation

$$rf(s_t)dt = \mathrm{E}_t^Q[df] ,$$

and application of Ito's lemma to its RHS.

(24) $$f(s_t) = Bs_t^{\beta},$$

where $\beta \equiv \beta_2 > 1$ and $B \equiv B_2$ is the constant of integration associated with this characteristic root.

Private holders of seigniorage and government debts are eager to get a return on their portfolio holding that depends on seigniorage issuance by the government[10]. However, if the latter responded with unrestricted seigniorage increase, the result of such a straightforward («inflationary») policy would be ambiguous, for investors are short in seigniorage and long in debts. By selling seigniorage and buying new debts investors relinquish compensation they received for seigniorage holding, and this no-arbitrage process is governed by Eq. (22). Given the constant parameters of the system, the government, servicing its debt at the continuously compounded rate $r > 0$, is obliged to sustain the riskless rate of seigniorage issuance:

$$r - \delta = a - \lambda\sigma$$

which is on terms with investors' expectations of their return on portfolio holding. The feasibility of this adjustment process being granted, the solution to (22) would provide the equilibrium strategy of government borrowing with respect to seigniorage issuance [21]. It calls for a policy coordination at least in this aspect, and monetary policy to be efficient has to take into account the short-run reactions of private investors on financial markets in the economy of transition.

Government debt market equilibrium

Rational behaviour of private investors being modeled as riskless replication and hedging made it possible to estimate the market value of debt (the steady state debt), $b(s_t)$, and the value of the opportunity to continue lending to the government, $f(s_t)$, at any point of time. We will follow the general framework of the *Black–Sholes–Merton model* [3].

Repeat again that the domestic government debt (or the macrodebt) is a risky asset, the riskiness of which is attributed to the non-zero probability of default[11]. The total market value of government debt (macrodebt) held by private investors at any moment of time $t < T$ is represented by a portfolio of the debt outstanding, $\exp\{-r(T-t)\}F$, where F is the *par* value, and the value of the opportunity to continue lending to the government, $f(s,t) = f(s_t)$:

(25) $$b(s_t) = D(s_t) + f(s_t),$$

where $D(s_t)$ is the market value of the debt outstanding[12]. In means that being sold out at a price, $b(s_t)$, total debt is equal to the value of the debt outstanding (sunk cost to the lenders) and to the value of the opportunity to continue lending to

[10] Since portfolio holders hedge their positions the rate of return might be considered to remain constant.

[11] There is no interest rate risk since parameters of the system are constant.

[12] For perpetual debt and borrowing the discount factor is just unity.

the government. This opportunity is imbedded into the financial structure of the economy of transition and lenders acquired it while extending their loans to the government. Usually «new» loans (debts to the government) are of short-maturity in the economy of transition.

Rational investors are expecting to get market value, $b(s_t)$, in exchange for the face value F of the debt outstanding and current loans, $f(s_t)$, to be extended under some conditions. Assuming these loans to be subordinated to the debt outstanding, lending to the government becomes analogous to a (plain vanilla) *call option* on seigniorage as its underlying. Private investors (or option holders) have the right to lend the sum of $f(s_t)$ to the government while the latter is obliged to redeem its debt at face value, F. Due to the fact that lending can take place at any time, it becomes similar to an American perpetual call option:

$$f(s_t) = \max\{b(s_t) - F, 0\}, \tag{26}$$

where the face value of a debt outstanding, F, is interpreted as the strike price of an option. Since macrodebt is risky, the holders of the debt outstanding, in their turn, are expecting to receive the market value

$$D(s_t) = F - \max[F - b(s_t), 0], \tag{27}$$

which is the *par* value less losses incurred by default. These losses are given by the put-to-default option:

$$P(s_t) = \max[F - b(s_t), 0].$$

The latter represents the value of guarantees to the debt holders to make riskless the debt outstanding. Since the market value of debt outstanding is given by

$$D(s_t) = F - P(s_t), \tag{28}$$

the equilibrium on the domestic government debt market is to satisfy the following equation:

$$b(s_t) + P(s_t) = f(s_t) + F. \tag{29}$$

Eq. (29) is the *put-call equivalence* on the market of a risky macrodebt for put and call options to have the same exercise price. In terms of costs and returns to the portfolio holders the equilibrium (29) can be expressed as follows. From the point of view of investors their cost of lending to the government consists of two parts: debt outstanding (or sunk costs) and value of the (imbedded) option to continue lending. In equilibrium these costs have to be equal to expected revenues: market value of total debt and value of debt guaranties. For the simple binomial case the no-arbitrage equivalence of costs and returns is schematically represented (for European options) in the table 1.

The Black–Sholes formulas for call and put options (with a correction due to (5))[13] can be used in order to evaluate amounts of lending and debt depreciation. The perpetual American call option (24) is a particular case of the Black–Sholes option pricing formula.

[13] We treat the total debt as the underlying on which options are written.

Table 1.

Position	Current time, t	Time of maturity, T	
		$b(s_T) > F$	$b(s_T) \le F$
Government debt outstanding	$-F\exp\{-r(t-T)\}$	$+F$	$+F$
Value of the opportunity to continue lending	$-f(s_t)$	$b(s_T)-F$	0
Buy market value of debt	$-b(s_t)$	$+b(s_T)$	$+b(s_T)$
Buy guaranties	$-P(s_t)$	0	$F-b(s_T)$
Costs and returns	$F\exp\{-r(t-T)\} + f(s_t) =$ $b(s_t) + P(s_t)$		

One point of seigniorage issuance, namely, $s_t = \tilde{s}$, where the equality $f(\tilde{s}) = P(\tilde{s})$ takes place, is of particular interest. At this point the market value of debt outstanding is equal to its *par* value[14]. In other words, without external guaranties the government should stop to borrow in order to avoid default on its debt outstanding[15]. Being as simple as that, this suggestion is not feasible because borrowing in practice provides cash necessary for the government to roll the debt over.

Equilibrium Eq. (29) defines the condition of debt sustainability which is, due to cost-revenue equivalency:

(29') $$f(s_t) + F = b(s_t) + P(s_t),$$

depends upon the existence of an external (or international) guarantor. the economy of transition If the latter exists, then for the government borrowings: $0 < f(s_t) \le P(s_t)$, debt is sustainable. Conversely, for the positive put-to-default option, the absence of the external (or international) guarantor, who would stand ready to compensate the debt depreciation, $P(s_t) = 0$, transforms equilibrium (29) into disequilibrium:

(30) $$b(s_t) < f(s_t) + F,$$

and default on macrodebt is imminent. Inequality (30) can be interpreted differently. Since at seigniorage s_t value of the option to lend is higher that the debt premium, $b(s_t) - F$:

(31) $$f(s_t) > b(s_t) - F,$$

14) For the European options with common exercise price at maturity at the point s_T both call and put are at-the-money and $b(s_T) = F$.

15) Following H. Varian point $s_t = \tilde{s}$ could be called as the Polonius point («Neither a borrower, nor a lender be...», W. Shakespeare, *Hamlet*).

it is rational for investors not to lend but wait until seigniorage would increase. Thus the debt sustainability requires a proper assessment of seigniorage issuance that implies solving of the optimal stopping problem [7].

Optimal value seigniorage and sustainable debt

Due to economic considerations, normally, the value of the opportunity to lend (or to borrow for the government) is no more than the value of the debt outstanding (upper boundary) and no less than a debt premium (lower boundary):

$$b(s_t) \geq f(s_t) \geq b(s_t) - F \,, \tag{32}$$

where F is the face value of the debt outstanding.

Rational investors, making a loan to a government, are expected to exercise the option to lend, $f(s_t)$, when the debt premium, $b(s_t) - F$, is positive. But they are not obliged to provide the government with loanable funds at any value of seigniorage: they can wait or, in a sense, they are able to «maximize seigniorage». It would be rational for them to exercise the option where the debt premium is not just positive but maximal. In other words, in the range of feasible seigniorage values

$$0 \leq s_t \leq s^*$$

private investors are waiting (or are not lending) until seigniorage reaches its maximal value, $s_t = s^*$. This is equivalent to solving of equation:

$$r\Phi(s_t)dt = -\delta s_t f'(s_t)dt + \mathrm{E}_t^Q[d\Phi] \,, \tag{33}$$

which is the transformation of Eq. (19) with respect to portfolio (14) held by investors. Eq. (33) is *the Bellman equation* for the maximization of the net payoff to the risk-free portfolio (14). Thus the risk neutral valuation problem is represented as the optimal stopping problem.

The solution of Eq. (33) for the perpetual call option, $f(s_t)$, subject to the boundary condition:

$$b(s^*) = f(s^*) + F \,;\; P(s^*) = 0 \,, \tag{34}$$

gives the optimal value seigniorage, s^* [7, 15] at which the equivalence condition (29) reduces to (34). It means that guaranties are not paid out at s^*, because investors receive the full value of the debt outstanding and new loans.

The optimal value of seigniorage for an American perpetual call can be found as a solution of two simultaneous equations, evaluated at the point of maximum, s^*:

$$\begin{aligned} f(s^*) &= b(s^*) - F \\ f'(s^*) &= b'(s^*) \end{aligned} \,. \tag{35}$$

Substituting (5) and (24) into system (35), we get the optimal seigniorage, s^*:

$$s^* = \frac{\beta}{\beta - 1}\delta F \,\cdot \tag{36}$$

Thus, if government issued seigniorage at the amount of s^*, then it is able: to support the steady state debt $b(s^*)$ thus honouring its debt obligations; and to continue borrowing without debt guaranties since $P(s^*) = 0$.

At the optimal point, s^*, seigniorage is amplified with regard to *the point of Polonius*. The latter determines the common exercise price of call and put options:

(37) $$\frac{1}{\delta}s - F = 0,$$

and amount of seigniorage:

(38) $$\tilde{s} = \delta F$$

supports *al pari* debt outstanding, F, without new government borrowing or debt guarantees; both call and put options are *at-the-money*.

Due to the seigniorage amplification by $q = \frac{\beta}{\beta - 1}$, at the point of optimum, $s^* = \delta\,[F + f(s^*)]$, seigniorage becomes sufficiently large to secure redemption in full of the debt outstanding. All other points: $\tilde{s} \le s_t < s^*$ are feasible but not optimal because

(39) $$f(s_t) > b(s_t) - F \text{ for } \forall s_t \ne s^*,$$

and rational holders do not exercise the option to lend the government.

The potential consequences of seigniorage issuance have to be fully anticipated in the domestic monetary policy implementation. At the point s^* private investors are content with lending to the government because they receive maximal market value of a risky asset, $b(s^*)$, which is equal to the face value of the debt, F, plus debt premium. The government, in its turn, is able to borrow maximum of sums available while the guarantor economizes on the guarantees to default.

A Primer: Was the August, 1998 Default Inevitable?

As an example of the model application let us consider the Russian default on domestic debt in August 1998. The model was estimated on empirical data of Russian economic performance in the period of 1994–1999. The crucial assumption of the model evaluation was the existence of empirical proxy for the riskless rate of interest and the market price of risk[16)]. By no means evident, such an assumption could have been taken as a rough approximation to the reality of a transition economy. Hence all the results of our analysis of the Russian debt default are contingent to the model validity.

Empirical values of debt and seigniorage, the seigniorage drift and variance for the period of 1994–1999 were calculated using the data in [18]. The expected rate of return on government debt[17)] was estimated on data taken from [22] as ave-

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[16)] A zero-beta portfolio might exists in this case being priced so as to provide an expected return equal to the risk-free rate.

[17)] The annualized Treasury bill rate in 1997 [4] appeared to be very close to the parameter used in the model. Being a mere coincidence, it indicates, though indirectly, the possible range of the actual and the «risk-free» interest rates to be used in the model estimation.

rage of yield to maturity. Given the data, parameters of the model were estimated on the quarter basis as follows:

The expected risk-adjusted rate of return on government debt, $\mu = 0{,}094$;

The expected rate of coupon payments, $\delta = 0{,}053$;

The expected rate of seigniorage issuance, $\alpha = 0{,}041$;

The riskless rate of return on government debt, $r = 0{,}049$;

The volatility of seigniorage, $\sigma = 0{,}289$.

For the given above parameters the market price of risk is λ=0,156, risk premium is $\lambda\sigma$= 0,045, and the risk neutral rate of seigniorage growth is $\alpha-\lambda\sigma = r-\delta = -0{,}004$.

A. Let us estimate the amount of debt that could be supported by seigniorage issuance on the actual level.

Actual seigniorage issuance in nominal terms amounted to *R*74,2*bn* in 1998, implying its amount in the second quarter to be approximately around *R*18,55*bn*[18]. Thus, according to Eq. (7) the theoretical market value of total debt (its stationary value) for the *Q*3, 1998 was estimated at the level of $b(s_{98}) = R350bn$. The linear approximation to the value of a call and put options [2]:

$$f(s,t) = 0{,}4b(s,t)\exp\{-\delta(T-t)\}\sigma\sqrt{(T-t)}\,, \tag{40}$$

was used in order to calculate amounts of borrowing and the risk-to-default premium. These approximations indicated the respective sums as *R*20,5*bn* for the *Q*3, 1998, meaning that the government could have borrowed no more than that, had it found some external guarantor ready to extend the credit to the same amount.

Assuming that the debt outstanding had to be redeemed at full[19] in the third quarter of 1998 (due to the crisis occurence in August) the equilibrium equation (29) gives the following magnitudes:

$$R350bn + R20{,}5bn = F + R20{,}5bn\,. \tag{41}$$

Eq. (41) indicates that theoretical values of the «old» debt at *par* as $\widetilde{F} = R350bn$, and its current value, $D(s_{98}) = R329{,}5bn$. It might have the following interpretation: had seigniorage been issued in *Q*3, 1998 in amount of *R*20,5*bn*, the market value of the total debt could have been supported at *R*350*bn*. Meanwhile the government had to pay out *R*370,5*bn* of the «old debt» and «new» borrowings due in the same *Q*3 (recall that government «old» and «new» debts were composed mainly of three-month T-bills). Thus the assets appeared to be short of liabilities at *R*20,5*bn* and the government was forced to default.

Empirically, the market value of government debt in August 1998 was *R*231,8*bn* while the net borrowing in the second quarter amounted to *R*50*bn*.

The results thus obtained might be interpreted in the following way. By comparing the actual and the par value seigniorage, it may be concluded that the for-

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[18] In fact, on the quarter basis seigniorage issuance was virtually zero in the *Q*3, 1998 and even negative in *the Q*2, 1998 thus suggesting the emergence of the deep rooted financial distress long before the August, 1998. Recall, that the model is subject to inequality $0 \le s_t \le s^*$, and the negative seigniorage was ruled out.

[19] Note, that the actual debt service considerations are determined by the same factors as in Eq. (29) subject to additional factor of the risk-adjusted rate of borrowing, $\mu = \mu(t)$.

mer appeared to be too small to support the domestic debt floating *al pari*. In July-August 1998 Bank of Russia even decreased the seigniorage issuance by $R16{,}4bn$, and that action just aggravated, in our opinion, the troubles with the domestic debt servicing (whether it had helped to support the crippled foreign exchange rate remained doubtful, as well). The monetary policy of the Russian central bank during that period was extremely tight. Being preoccupied with the fight against inflation it did not supply enough liquidity, and the real sector of the Russian economy was forced to rely heavily on barter accompanied by mass and persistent arrears. Quite naturally, the extremely tight monetary policy had its implications on the debt service, and to some extent Eq. (33) which became the inequality in effect, reflected this fact. Hence too small seigniorage issuance could be considered as at least one of the causes of the default on the domestic debt.

Judging from Eq. (33 – that made possible the debt rollover) the government had the following alternatives[20)] to avoid default:

a) it could have found an external guarantor of its debt ready to loan $R20{,}5bn$, or equivalently, about $3*bn*;

b) it could have settled an arrangement with lenders about the «new debts» restructuring.

Both alternatives, in our opinion, would have given the government some breezing time, and the crisis could have been postponed until, say, October which is a favorite month for such kind of events.

The model suggests another strategy that would have excluded the mere possibility of default, at least in the short run. In order to «economize» on the debt guarantees or to make put-to-default option to be out-of-the-money, $P(s^*) = 0$, the issuance of seigniorage should be increased by factor of 2,5. Since the risk-neutral rate of seigniorage growth is very small ($r-\delta=-0{,}004$), the simpler form approximates the Black–Sholes equation (22):

$$\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 f}{\partial s^2} - rf(s,t) = 0, \tag{42}$$

with the following characteristic equation:

$$\beta^2 - \beta - \frac{2r}{\sigma^2} = 0. \tag{43}$$

The latter has the largest positive root $\beta_2 \equiv \beta = 1{,}68$, and the uncertainty amplifier, $q=2{,}47$. The latter required the increase of seigniorage at the rate of $s^* = R45{,}84bn$ in *Q*3, 1998. Were the equilibrium debt remained at $R350bn$; the new level of seigniorage would have supported much larger market value of debt, $b(s^*) = R400{,}6bn$ and new borrowing of $f(s^*) = R50{,}62bn$. These values satisfy the following equilibrium equation:

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[20] Theoretically, were the government appeared to be able to stop the borrowing altogether, the debt could have considered to be redeemable. But , quite evidently, it was not a feasible strategy: cash payments, that made possible the debt rollover, were possible only via borrowing continuation. Interesting enough to notice that calibration of the model indicates the level of the «break-even» seigniorage at $\tilde{s}_{98} = \delta \tilde{F} = R18{,}55bn$ which was very close to the amount of borrowing analyzed above.

(44) $$R400{,}7bn \cong R350bn + R50{,}62bn\,,$$

which requires no risk-to-default compensation on behalf of the external guarantor.

Notice, that due to crudeness of linear approximations the larger seigniorage issuance determines the larger coupon rate, δ^*=0,1144, estimated via calibration of the model. This parameter adjustment reflects the inflationary pressure, which accounts for the increases in the rates of seigniorage issuance, borrowing and risk compensation. Though the model does not allow for the monetary causes of inflation its parameters have to be adjusted were seigniorage grows too fast.

B. Let us estimate now the amount of seigniorage required to support the actual debt outstanding *al pari*, $F = R436bn$, on the eve of the default declaration.

We start at condition (37) and thus are guaranteed that theoretical debt is always positive[21]. The debt *al pari* requires seigniorage of $R23bn$ in order for equilibrium equation (34) to be satisfied. Since the actual face value of the debt was larger than the previously estimated «theoretical value», $\widetilde{F} = R350bn$, the required and the optimal seigniorage in this case should be larger too. The latter amounted to $s^* = R57{,}1bn$ with the market value of debt $b(s^*) = R499{,}1bn$ and borrowing, $f(s^*) = R63{,}1bn$. The equilibrium equation (34) at the point of the optimal seigniorage issuance, s^*, in this case is as follows:

$$R499{,}1bn{=}R436{,}0bn{+}R63{,}1bn,$$

with the same rate of the convenience yield, δ^*, as before. Thus, in our opinion, the increase of seigniorage could have greatly helped to the solution of the debt problem without rejecting the possibility to continue its rollover.

The average of our two estimates of the optimal seigniorage (on the yearly basis) yields $R205{,}9bn$. Had seigniorage been increased to this amount, a much higher level of debt outstanding (steady state debt) could have been sustained, implying the avoidance of the debt default. Interestingly to note, that theoretically suggested level of seigniorage in $R205{,}9bn$ seems to be too large, but it is only at the first glance. The actual increase in M2 money aggregate had reached the magnitude of about $R200bn$ in the period of August 1998 to May 1999, and it was amounted to $R108{,}9bn$ in September-December 1998, just in the aftermath of the crisis.

The proper timing for the seigniorage issuance was, definitely, of crucial importance. Very little could have been done in August 1998. The increase in the money supply should have been started well ahead of the default, when the bells of the Asian financial crisis had tolled for the first time in October 1997. If the government had increased seigniorage issuance, that was strongly coherent with the economy requirements of that time, it could have avoided the default on its domestic debt. The latter resulted not from the «debt roulette» *per se*, but rather from the exposed inability, both objective and subjective, of Russian authorities to play it properly. These considerations conform in general conclusions being expressed in [9] regarding government borrowing in the Russian economy of transition.

[21] Though being ruled out theoretically in the model, seigniorage occurred to be negative on the monthly basis.

Of course, some price had to be paid, namely the increase in domestic inflation, for the implementation of the proper policy mix. By no means praising inflation, in our opinion, it seemed to be much lesser burden for the economy than the debt default. The «easy money» policy seemed to be rather appropriate recipe for the transition economy of Russia in 1998, with its sharply imbalanced money market, deep demonetization of the economy, huge and persistent arrears.

Some concluding remarks

The model demonstrated the deficiency of the extremely «tight money» policy in the case of the Russian debt default and thus calls for the better policy coordination. The lack of coordinated fiscal and monetary policy brought about not only revival of barter, which is the worst form of exchange, but the government inability to honour its debt obligations, as well. Both of them undermined economic reforms, which was rather evident from the Russian experience. In this aspect the diffusion-type model was able to produce some reasonable insights about the process underlying.

The design of the proper policy mix has to be based upon clear understanding of the monetary effects upon the debt dynamics and managing of the debt service. The model demonstrated a feedback loop between seigniorage and government debt and borrowing that was more complex than its deterministic alternative. Contrary to a simple deterministic dichotomy: more money – less debts, and *vice versa*, in the stochastic environment, given the hedge ratio, an increase of seigniorage brought about higher government borrowing, and thus higher market value of the debt outstanding[22]. New debts substituted to some extend for new money by rational investors who were able to assess properly market risks and hedge their portfolios of seigniorage and new loans against losses. Shorting of seigniorage made investors to be content with government borrowing which they support by purchasing new debts. This process has to be facilitated by a coordinated monetary and fiscal policy that in the short run made it possible to hold riskless portfolios. The proposed strategy of amplifying seigniorage was an impulse rather than a sustainable increase in the rate of money growth, and in the (very) short run inflationary pressure seemed to be minimal being measured by the riskless rate of seigniorage growth, $r-\delta$.

The requirement of decomposition of the risk-adjusted rate of return on the financial market of the economy of transition was much stronger than the ordinary requirements of rational expectations. In the case of the stochastic debt accumulation the former called for Eq. (4), which could be transformed into

$$\mu\, b(s_t)dt = s_t dt + as_t b'(s_t) + \frac{1}{2}(\sigma\, s_t)^2 b''(s_t) \quad (45)$$

Evidently, the second-order ODE (45) can be solved given the parameters of the risky debt market but it would have given us no clues as to the borrowing strategy, which was modeled by the call option $f(s_t, t) = f(s_t)$ and required the riskless portfolio of «new» debts and seigniorage.

The problem of the adequacy of geometric stochastic process is far from being resolved. In our case, *a priori* approach based upon the diffusion-type considerations

[22] In the process of continuous portfolio adjustments the hedge ratio is changing itself that complicates the process even further.

was undertaken though it was far from being evident that the GSP modeled by Eq. (1) was the best hypothesis that fits the data. The most unpleasant thing in this respect was the occurrence of negative seigniorage in the monthly data of Russian M2 money aggregates. Though this inconvenience can be tackled with even in our simplest model, it nevertheless casts some doubts upon the validity of the hypothesis of the geometric stochastic process standard in financial economics.

From the model standpoint it is nonsense, for $s_t < 0$ means that the government being a borrower yet receives its money back instead of spending it out as coupon payments. For that reason, negative seigniorage was ruled out by the imposition of inequalities $0 \le s_t \le s^*$. The violation of them (the nonexistence of absorption in the case of debt) could have considered as a manifestation of «bounded rationality» [20]) associated with the early stages of the economic transformation.

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