Competitive Storage and Commodity Price in Continuous Time

Georgy Trofimov

Laboratory for Modern Financial Studies,
141-1, village Zhukovka, Odintsovsky district, Moscow region, 143082, Russian Federation.
E-mail: g.trofimov@ifs.ru

The competitive storage model is analyzed in the literature in discrete time and is applied for empirical studies of the markets for agricultural commodities. In this model, there is a storable commodity supplied at every period with stochastic disturbances, and there are traders who aim at making speculative profits by using their storage. The literature has established results such as the existence of an equilibrium price function, which depends on the current availability of the commodity. The present article proposes an extension to the competitive storage model by considering continuous time. The relevance of this extension is justified by the technical convenience it brings, as well as by its suitability to the mineral markets in which the time series data is available for daily frequency. I consider serially correlated disturbances of net supply together with an upper bound on storage capacity, and characterize the equilibrium price function in this framework. The no-arbitrage and no-trade conditions define the intertemporal choice of commodity traders and imply the existence of an equilibrium price function. The equilibrium price depends on "the long-term availability of commodity" defined as the sum of storage and the expected cumulative disturbances of net supply over the infinite horizon. The various cases of the equilibrium price dynamics, such as "full storage", "empty storage" and "the trading zone", are characterized. The two types of the equilibrium price function, which are relevant to full storage, are revealed, and an explicit approximate solution for the case of a low-elastic net demand is derived. Numerical simulations of the equilibrium price function demonstrate the effects of the model parameters on this function.

The author is thankful to Alexey Pomansky, Sergey Drobyshevsky and the anonymous referee for valuable comments.

Georgy Trofimov – Candidate of economic sciences, the chief researcher of Laboratory for Modern Financial Studies.

The article was received: 01.09.2022/The article is accepted for publication: 27.10.2022.

Key words: commodity price; speculative trade; storage capacity; long-term availability; equilibrium price function; boundary conditions.

JEL Classification: Q02, C61, D84.

DOI: 10.17323/1813-8691-2022-26-4-523-551

For citation: G. Trofimov. Competitive Storage and Commodity Price in Continuous Time. *HSE Economic Journal.* 2022; 26(4): 523–551.

1. Introduction

The competitive storage model is a well-established theoretical tool for studying the price behaviour of storable commodities. The model actors, apart from commodity consumers and producers, are competitive risk-neutral traders who aim at making speculative profits by using their storage. Intertemporal storage arbitrage resulting from this trade generates the model price behaviour consistent with the observed qualitative features of commodity prices. These features include positive serial correlation, skewness of distribution, succession of long periods of doldrums and short periods of high or low prices. Positive autocorrelation of prices is explained by the smoothing effect of storage arbitrage that can buffer supply and demand shocks.

Beginning from Gustafson's (1958) major contribution to the study of the influence of storage on commodity price volatility, competitive storage models are formulated in discrete time. This is justified by the focus of empirical research based on this model on the prices of agricultural commodities supplied annually with stochastic disturbances. In the base model of commodity storage offered by Gustafson, production is exogenous and given by annual stochastic harvests that are serially uncorrelated. The annual price of agricultural commodity is formed after realization of harvest, and annual decisions on commodity storage are based on the annual prices observed by traders.

However, solutions of the discrete-time storage models involve analytical difficulties caused by the non-negativity constraint on storage. Traders cannot "borrow" a commodity from future harvests, and stock-outs occur in the model under prices above a threshold level. The stationary rational expectations equilibrium is defined by a no-arbitrage condition implying a mapping of price into the next-period price expectation. Because of the non-negativity constraint and a feedback effect of price on consumers-producers' net demand, such a mapping implies a highly non-linear functional equation on price that cannot be linearized and solved analytically. Gustafson (1958) proposed a numerical method for solving the storage model. Since then, the sophisticated computational techniques have been developed for estimating the model parameters and numerical solutions by Wright (1991), Deaton and Laroque (1992, 1995, 1996), Cafiero et al. (2011), Guerra et al. (2015), Oglend and Kleppe (2017) and other authors.

This paper develops a different approach, proposed in the monograph by Vavilov and Trofimov (2021), to the formal analysis of the competitive storage model in continuous time. The motivation of this approach is twofold. First, applying stochastic calculus under the assumption of a continuous-time stochastic process on supply or demand side facilitates formal analysis due to the possibility of linearization of the expected price growth differential. The second motive is the relevance of the continuous-time model of competitive storage to mineral commodity prices, for which the data series with very short time periods can be a subject of empirical studies. For example, the daily data series are available for the West Texas Intermediate (WTI) crude oil prices. A time-continuous storage model can be applied for empirical analysis of the high-frequency data series of commodity prices.

The model of this paper rests, to some extent, upon the extensions of the base discrete-time model of annual commodity storage with serially uncorrelated supply shocks and unlimited storage capacity. An important extension was the introduction of positive serial correlation of supply or demand shocks. Chambers and Baily (1996) generalized the results of Deaton and Laroque (1992) for the base model by demonstrating the existence of a unique stationary rational expectations equilibrium for the model with autocorrelated supply process. Deaton and Laroque (1995, 1996) showed that the price autocorrelations, as implied by their econometric estimation of the extended model, are significantly lower than those observed for the series of actual commodity prices. Guerera et al. (2011, 2015) rejected this inference by using a finer approximation of the model solution. Dvir and Rogoff (2009) applied a discrete-time storage model with serially correlated and persistent demand shocks for the analysis of the crude oil price volatility.

Oglend and Kleppe (2017) extended the base storage model by introducing a storage capacity constraint to capture negative price spikes that can occur under supply gluts. They assumed that speculative storage is bounded from above and demonstrated that a sequence of price functions obtained by the fixed-point iterations of the price mapping converges to the equilibrium price function.

Stochastic dynamics of storable commodity prices in continuous time have been examined in the literature on quantitative finance. For example, Ribeiro and Hodges (2004) extended Schwartz's (1997) model of commodity futures pricing based on a joint two-factor stochastic process for commodity spot price and convenience yield that are given exogenously. In models of this class, introduced by Gibson and Schwartz (1990), commodity traders' behaviour is not considered and storage can influence a commodity price only implicitly, through the convenience yield obtained by producers or consumers from holding storage. However, in a recent paper, Karimi et al. (2021) analyse in a continuous time framework a modification of the base Deaton and Laroque (1992) model of competitive storage with serially uncorrelated supply shocks and unlimited storage capacity. Their model solution is given by the equilibrium price function of a variable defined as commodity demand and represented as an endogenous stochastic process.

The continuous-time storage model in the present article is based on more general assumptions. It features both an intertemporal dependence of net supply disturbances represented by a mean-reverting stochastic process and a capacity constraint on storage, in addition to the non-negativity constraint. The model with unlimited storage capacity is considered as a special case.

The intertemporal equilibrium of the model is defined by the zero-profit condition of no-arbitrage for the interior solution in *the trading zone* and by the negative-profit conditions of no trade for the corner solutions. Under *empty storage*, the non-negativity constraint on storage is binding, and the no-trade condition requires the expected profit from buying a commodity to be negative. Under *full storage*, the capacity constraint on storage is binding, and the no-trade condition is that the expected profit from selling a commodity is negative. The no-trade conditions, thus, ensure that traders have no incentives to buy under empty storage or to sell under full storage.

The solution for a stationary rational expectations equilibrium of the model is derived analytically in this paper, unlike the mentioned above monograph by Vavilov and Trofimov

(2021)[1]. This solution is given by an equilibrium price function of *the long-term availability* defined as the sum of storage and the expected cumulative disturbances of net supply of commodity over the infinite horizon. In the discrete-time competitive storage model with serially independent supply shocks, the state variable is *the current availability* defined as storage by the beginning of year plus annual harvest (e.g. [Gustafson, 1958; Deaton, Laroque, 1992]). In the discrete-time models with serially correlated supply shocks, there are two state variables: storage and annual harvest (e.g. [Chambers, Bailey, 1996]).

For the continuous-time model, the zero-profit condition of no-arbitrage implies, due to linearization, a second-order non-linear ordinary differential equation for the equilibrium price function. I demonstrate the existence and uniqueness of this function under the free-boundary conditions of value matching and smooth pasting. By using an approximation for this equation for a low-elastic net demand, I derive an explicit solution satisfying these boundary conditions. The equilibrium price function is given by a linear-quadratic combination of exponential functions of the long-term availability, which is derived for a bounded storage capacity and for the unbounded one as the limit case.

For a bounded storage capacity, there exist two types of equilibrium price function. For the first type, the regime switching between speculative trade and full storage with no such trade occurs at a positive price. For the second type, this switching takes place at zero price coinciding with the kink point of a piecewise-linear inverse net demand function assumed in the model.

The second type of equilibrium solution is relevant to the empirical evidence on negative commodity price spikes. The crude oil oversupply shocks that occurred in the second quarter of 2020 provide an example of the influence of a bounded storage capacity on the crude oil price dynamics. In April 2020, commercial storage in the United States approached to the level of full utilization, while the WTI crude oil price dropped to zero.

With the analytical solution of the model, a series of numerical simulations were conducted for the two types of capacity-constrained solutions and for the case of unbounded storage capacity. The equilibrium price functions are calculated under various values of the model parameters. In particular, simulations demonstrate the effects of storage capacity and the variance of net supply disturbances on the commodity price level and on the boundaries of the trading zone, which is defined by the interval of the state variable where speculative trade takes place.

2. The Model

Consumers and producers of a commodity in the model generate a net demand covered by competitive traders who own storage and get speculative profits from commodity trade. We consider, first, a commodity market without traders and then introduce these participants.

2.1. Commodity spot market

The instantaneous net demand is represented as the difference between a price-dependent non-stochastic term called *the net demand function* $y(p)$, where p is the commodity spot

[1] In the early version of this model, Vavilov and Trofimov (2021) did not use the no-trade conditions in the description of commodity traders' behaviour and considered a different case of net demand function. They characterized the model solution, but did not derive an analytical solution for the equilibrium price function and did not conduct numerical simulations.

price, and an exogenous stochastic variable x that defines instantaneous *net supply disturbances*. Here and henceforth, we do not use a time index in the model notations if it is not necessary.

The instantaneous net demand function $y(p)$ is defined for all $p \geq 0$, continuously differentiable, decreasing and convex. This function is positive for low prices and negative for high ones. An example of this function that will be used in what follows is the linear net demand shown in Fig. 1a: $y(p) = b - \delta p$, with $\delta > 0$. The linear net demand function can represent the difference between inelastic demand $D(p) = b_d$ and the linear supply function $S(p) = b_s + \delta p$, so that $b = b_d - b_s > 0$. The magnitude of parameter δ is small for a low-elastic supply.

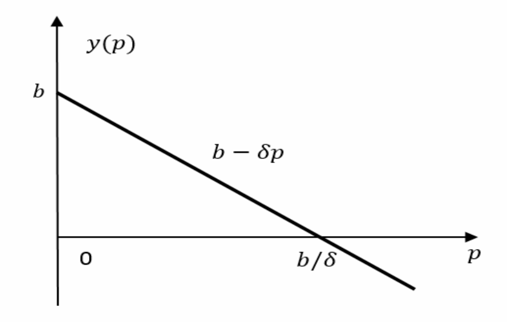

Fig. 1a. Net demand function

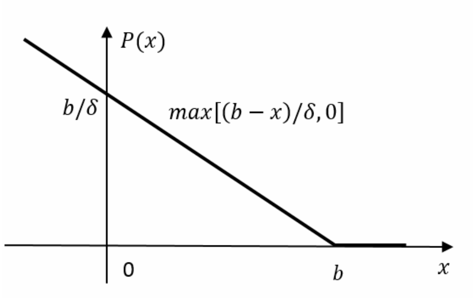

Fig. 1b. Market-clearing price

The net supply disturbances x follow the mean-reverting process:

(1) $$dx = -\mu x dt + \sigma dw,$$

where $\mu > 0$ is the rate of mean reversion, dw is an increment of the standard Wiener process with long-term mean 0 and instantaneous variance 1, and σ is the instantaneous standard deviation of net supply shocks σdw. The variance of these shocks depends linearly on the length of an infinitesimal time interval, $E(\sigma dw)^2 = \sigma^2 dt$. The stochastic equation (1) means that in any such interval $(t, t + dt)$, any small change of net supply dx results from the drift to the long-term zero mean, $-\mu x dt$, and the impact of serially uncorrelated shocks, σdw.

In the absence of speculative trade, the market clears at any instant, implying that

(2) $$y(p) = x.$$

The market-clearing price satisfying (2) is the inverse net demand function of stochastic disturbances denoted as $P(x) \equiv y^{-1}(x)$. For the linear net demand function $y(p) = b - \delta p$, the market-clearing price is $P(x) = (b - x)/\delta$ if $x \leq b$. The price is zero under oversupply: $P(x) = 0$ if $x > b$. Hence, the market price in the absence of storage is the piecewise-linear function of net supply disturbance:

(3) $$P(x) = \max\left[(b - x)/\delta, 0\right].$$

The function $P(x)$ is shown in Fig. 1b. It is defined for all real numbers and intersects the vertical axis at point $P(0) = b/\delta$, which is the market-clearing price for the long-term mean of net supply process (1). The market cannot be cleared under oversupply because the volume of excess supply $x-b$ cannot be sold at a positive price and is removed from the market in the absence of storage.

2.2. Traders and storage

Suppose that competitive homogenous traders are present in the commodity market and consider a continuous-time trader's problem. A risk-neutral trader maximizes the expected discounted cash flow from trade over an infinite time horizon:

$$E_0 \int_0^\infty e^{-rt} p_t q_t dt, \tag{4}$$

subject to the storage balance equation:

$$ds_t = -q_t dt, \tag{5}$$

and the two-side constraint:

$$0 \le s_t \le \overline{s}, \tag{6}$$

where r denotes the riskless real interest rate, s storage, q the intensity of trade and $\overline{s}$ the storage capacity. Storage is costless and does not deteriorate in time. At any instant, traders observe the current price p and choose the intensity of trade q. They are commodity sellers if $q > 0$ and buyers if $q < 0$. For a small time interval $(t, t+dt)$ the volume of trade is qdt and the cash flow is $pqdt$. According to (5), the change of storage ds equals the volume of trade, $-qdt$. According to (6), storage is constrained from below by zero and from above by storage capacity. The initial storage is $s_0 \ge 0$.

At any instant, traders clear the market:

$$q = y(p) - x. \tag{7}$$

Thus, equations of the model with traders include the trader's problem (4)–(6), the market-clearing condition (7) and equation (1) specifying the process for net supply disturbances.

3. The Intertemporal Equilibrium Conditions

The equilibrium conditions of trade or no trade relate to the expected rate of price growth at any instant and depend on whether or not the storage constraints (6) are binding.

3.1. Trading zone

Trade with the use of storage occurs if both constraints (6) are non-binding, $0 < s < \bar{s}$. There are no arbitrage opportunities in the trading zone implying that the expected growth rate of price is equal to the interest rate:

(8) $$E_t dp = rpdt.$$

The expected net profit from trade at time t is zero, and the equilibrium dynamic of commodity price is governed by the stochastic Hotelling rule (8).

3.2. Empty storage

Under empty storage, $s = 0$ and the no-trade condition is that the expected rate of price growth is below the interest rate:

(9) $$E_t dp < rpdt.$$

On the one hand, traders cannot sell the commodity because the stocks are absent. On the other hand, they do not buy because the expected rate of price growth is below the interest rate (it is better to deposit money in bank than to take a long position in commodity). Since there is no trade under empty storage $(q = 0)$, the price fully absorbs any net supply disturbance x:

(10) $$p = P(x).$$

For the linear net demand function, $y(p) = b - \delta p$, the price under empty storage is given by (3).

3.3. Full storage

Under full storage, $s = \bar{s}$ and traders do not buy the commodity because they have no available storage capacity. They do not sell if the expected growth rate of price is above the interest rate:

(11) $$E_t dp > rpdt,$$

given that the price is positive, $p > 0$, or if the price is zero, $p = 0$. In the former case, the expected rate of return on storing exceeds the interest rate (it is better to store the commodity than to sell and deposit money in bank). In the latter case, the sale at zero price means a gift to consumers that does not make sense for traders. Thus, speculative trade is absent, $q = 0$, under condition (11) or zero price. The price under full storage fully absorbs shocks, $p = P(x)$, and is given by (3) for the linear net demand.

4. Stationary Rational Expectations Equilibrium

In Deaton and Laroque's (1992) discrete-time model, the current availability means the amount of commodity available in any period to consumers and traders and defined as the sum of storage and supply per current period. Here *the long-term availability* of commodity is defined for any instant as the sum of storage and *the cumulative net supply disturbances* expected for the long term:

$$a_t = s_t + E_t \int_0^{\infty} x_{t+\tau} d\tau. \tag{12}$$

The term added to storage is the conditional expectation of net supply disturbances integrated over an infinite time horizon. For the stochastic process (1), the expected lagged net supply for any time moment $t+\tau$ is $E_t x_{t+\tau} = e^{-\mu\tau} x_t$. Integrating over time interval $0 \le \tau \le T$, we obtain: $E_t \int_0^T x_{t+\tau} d\tau = x_t \int_0^T e^{-\mu\tau} d\tau = x_t \left(1 - e^{-\mu T}\right)/\mu$. For $T = \infty$, this equals x_t/μ, and formula (12) is written as

$$a_t = s_t + x_t/\mu. \tag{12'}$$

The initial long-term availability is given by $a_0 = s_0 + x_0/\mu$, where x_0 is the initial net supply. The variable a_t can be negative if the net supply disturbance x_t is negative.

Consider a stationary rational expectations equilibrium, for which the stochastic net supply process (1) is stationary. We will show that the long-term availability defined as $a = s + x/\mu$ is the state variable sufficient to determine the equilibrium price. For the stationary equilibrium, consider the equilibrium price function of the long-term availability $p(a)$ defined for real numbers, non-increasing and twice continuously differentiable. Under speculative trade, this function satisfies the no-arbitrage equation (8) rewritten as

$$Edp(a) = rp(a)dt. \tag{13}$$

The expectation of price change for stationary equilibrium is taken with respect to the state variable a that must contain all information relevant to the current price determination. Here and henceforth, we omit this variable in the notation of conditional expectation in stationary equilibrium.

Under empty storage, $a = x/\mu$. Let us introduce, for the sake of notational convenience, the inverse net demand as the function of availability:

$$\overline{P}(a) \equiv P(\mu a) = \max\left[\left(b - \mu a\right)/\delta, 0\right].$$

The price under empty storage is always positive, hence $\overline{P}(a) = \left(b - \mu a\right)/\delta$. Under full storage, $a = \overline{s} + x/\mu$, and the inverse net demand is represented as

$$\underline{P}(a) \equiv P\left(\mu(a-\overline{s})\right) = \max\left[\left(b+\mu(\overline{s}-a)\right)/\delta, 0\right].$$

A *synthetic* equilibrium price function $\Psi(a)$ is defined for real numbers by combining the intertemporal equilibrium conditions (8) through (11) as

$$\Psi(a) = \begin{cases} \overline{P}(a), & \text{if } Ed\overline{P}(a) < r\overline{P}(a)dt \\ p(a), & \text{if } Edp(a) = rp(a)dt \\ \underline{P}(a), & \text{if } \underline{P}(a) > 0,\ Ed\underline{P}(a) > r\underline{P}(a)dt \\ \underline{P}(a), & \text{if } \underline{P}(a) = 0. \end{cases} \quad (14)$$

Conditions of storage balance (5) and market clearing (7) can be combined as

$$ds = xdt - y\left(\Psi(a)\right)dt, \quad (15)$$

implying that for any small time interval the change of storage covers net demand.

4.1. The equilibrium price function

Under speculative trade, the state variable a defined as (12′) is driven by the stochastic Ito process:

$$da = -y\left(p(a)\right)dt + (\sigma/\mu)dw, \quad (16)$$

that results from combining the net supply process (1) with storage balance under market clearing (15):

$$da = ds + dx/\mu = xdt - y\left(p(a)\right)dt + \left(-\mu xdt + \sigma dw\right)/\mu = -y\left(p(a)\right)dt + \left(\sigma/\mu\right)dw.$$

The process for long-term availability (16) has the non-linear drift rate $-y\left(p(a)\right)$ and the instantaneous standard deviation σ/μ. This is the standard deviation of the stochastic process for x *adjusted for autocorrelation*. The lower the rate of mean reversion μ, the higher the autocorrelation of x and the standard deviation for a, σ/μ. If μ is small, the net supply disturbances are persistent, and the instantaneous variance of long-term availability is large.

Under empty storage, $a = x/\mu$ and $da = dx/\mu$. The state variable evolves as:

$$da = -y\left(\Psi(a)\right)dt + (\sigma/\mu)dw, \quad (17)$$

because, from (1), $da = dx/\mu = -xdt + (\sigma/\mu)dw$ and, from (15), $x = y\left(\Psi(a)\right)$ for $ds = 0$. Similarly, under full storage, $a = \overline{s} + x/\mu$ and $da = dx/\mu = -xdt + (\sigma/\mu)dw$, hence the differential for a is given by (17) because $x = y\left(\Psi(a)\right)$ for $ds = 0$.

It is important that the net supply term x vanishes from the right-hand side of equation (16) or (17) and does not influence directly the differential da. The stochastic process for a incorporates conditions of equilibrium: the storage balance (5) and the market clearing condition (7) for net demand function $y(p)$ under net supply shock dw. Thus, the long-term availability a satisfying dynamic equations (16) or (17) is indeed the state variable containing all available information, which is sufficient for the formation of price expectations and the equilibrium price determination according to the intertemporal equilibrium conditions (14).

Applying Ito's Lemma for the stochastic process (16), one can express the expected price differential as the first-order series expansion involving the second-order derivative of the equilibrium price function (EPF):

$$Edp(a) = -p'(a)y\left(p(a)\right)dt + 0.5(\sigma/\mu)^2 p''(a)dt. \quad (18)$$

The expected price change results from the effect of net demand and the effect of the long-term availability variance.

Substituting the no-arbitrage condition (13) for the expectation term in equation (18), and dividing both sides by dt, yields the second-order non-linear differential equation for $p(a)$:

$$0.5\left(\sigma/\mu\right)^2 p''(a) - p'(a)y\left(p(a)\right) = rp(a). \quad (19)$$

The EPF $p(a)$ is a solution of this equation subject to the boundary conditions defined below. The non-linear term on the left-hand side of (19) captures the feedback effect of price on net demand, which is taken into account by commodity traders in their expectations of price change. The term in (19) with the instantaneous variance of long-term availability $\left(\sigma/\mu\right)^2$ can be significant in absolute size if μ is small – that is, demand-supply disturbances are persistent.

Thus, at any instant, the synthetic equilibrium price $\Psi(a)$ is determined by the state variable a that evolves according to the stochastic differential equation (16) or (17). The instantaneous change of storage is $ds = \left[x - y\left(\Psi(a)\right)\right]dt$, according to (15). The information about instantaneous net supply x is relevant to the change of storage under speculative trade but redundant *per se* for determination of the equilibrium price function.

4.2. Equilibrium price paths

Consider the price function $p(a)$ and the second-order differential equation (19) for the linear net demand $y(p) = b - \delta p$. One can represent this equation as the two-dimensional system of the first-order differential equations for $p(a)$ and its derivative:

$$p'(a) = z(a), \quad (20)$$

$$0.5(\sigma/\mu)^2 z'(a) = z(a)\left(b - \delta p(a)\right) + rp(a). \quad (21)$$

Equation (20) presents variable $z(a)$ as the derivative of equilibrium price with respect to the state variable, and equation (21) is identical to (19). The price function $p(a)$ and its derivative $z(a)$ are the phase variables determined by system (20), (21).

Fig. 2 depicts the phase plane of this system. Curved arrows show paths of price $p(a)$ and price derivative $z(a)$ under increasing argument a . We search for an EPF $p(a)$ and consider the fourth quadrant of the phase plane, where the price is positive and the price derivative is negative, since the equilibrium price function we are looking for should be non-increasing.

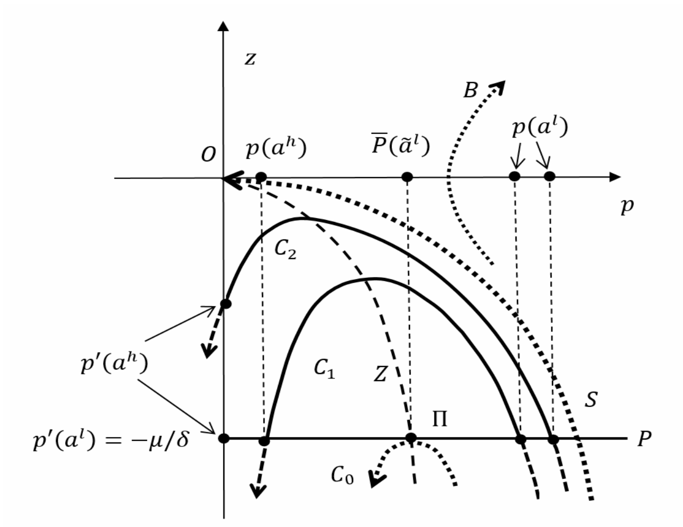

Fig. 2. The phase plane and the equilibrium price paths

The stationary state of system (20), (21) is the origin O because it is the intersection of the two loci, corresponding to zero derivatives: $p'(a)=0$ and $z'(a)=0$. The first one is line $z=0$, and the second one is dashed curve Z , the locus of zero second-order derivatives, shown in Fig. 2 and given by the function:

$$Z(p) = -\frac{rp}{b - \delta p}.$$

The characteristic equation of system (20), (21) linearized near the origin is

$$\begin{vmatrix} -\lambda & 1 \\ gr & gb - \lambda \end{vmatrix} = \lambda^2 - gb\lambda - gr = 0,$$

where λ is the characteristic root, $g = 2(\mu/\sigma)^2$. This equation has two real roots of a different sign, hence the stationary state is the saddle point.

As one can see in Fig. 2, there are several types of trajectories in the phase plane of system (20), (21). The saddle path S is drawn with the bold dotted curved arrow. It converges to the origin O as a tends to infinity, and it is the unique path in the fourth quadrant converging to the origin. The path drawn with the dotted curved arrow B is characterized by non-monotonic price dynamic. The price decreases with a for $z<0$ and, after intersecting the horizontal axis, $z=0$, it increases for $z>0$. The price growth under increasing availability is caused by a speculative bubble generated by perpetual storage accumulation, which is driven by expectations of further price growth. Thus, each *B*-type path represents a bubble solution.

We will focus on the types of trajectories C_1 and C_2 drawn in Fig. 2 with the bold curves. For these paths, the price decreases with a, and the price derivative is non-monotonic in a. As any path of type-C_1 or C_2 crosses curve Z (the locus of zero second-order derivatives, $z'(a)=0$), the price derivative $z(a)$ starts to decrease with a.

Horizontal line P in Fig. 2 depicts the locus of non-zero inverse net demand $\overline{P}(a)$ or $\underline{P}(a)$ in the phase plane. Each point of this locus is a combination of $\overline{P}(a)$, $\overline{P}'(a)$ or $\underline{P}(a)$, $\underline{P}'(a)$ for any a. For the linear net demand, locus P is given by the line:

$$z=-\mu/\delta$$

because $\overline{P}'(a)=\underline{P}'(a)=-\mu/\delta$ for positive prices. The path of type C_1 intersects the line of inverse net demand P twice, while the path of type C_2 only once.

The price paths of both types represent the EPFs under a limited storage capacity $\overline{s}<\infty$. As will be shown below, a price path belongs to type C_1 if $\overline{s}$ is relatively small and to type C_2 otherwise. The saddle path S is the limit case for type C_2 corresponding to the case of unbounded storage capacity, $\overline{s}=\infty$.

4.3. The boundary conditions

The points of intersection of the price paths of types C_1 and C_2 with the line of inverse net demand P are depicted in Fig. 2 as $\left(p\left(a^l\right), p'\left(a^l\right)\right)$ and $\left(p\left(a^h\right), p'\left(a^h\right)\right)$. These are the points of regime switching between empty storage, trading zone, and full storage that define the structure of the synthetic EPF $\Psi(a)$.

Fig. 3 portrays this function for the price path of type C_1, It is drawn with the bold solid curve and consists of three pieces. The first one, the EPF under trade $p(a)$ is a solution of the second-order differential equation (20)–(21). The function $p(a)$ is decreasing and convex-concave with the inflection point Z since derivative $z(a)$ for the price path of type C_1 depicted in Fig. 2 increases with a before the intersection of the locus of second-order derivatives and then decreases. The function $p(a)$ in Fig. 3 touches tangentially two other pieces of $\Psi(a)$ given by the functions of inverse net demand drawn as the kinked dotted lines: $\overline{P}(a)$ under empty storage and $\underline{P}(a)$ under full storage.

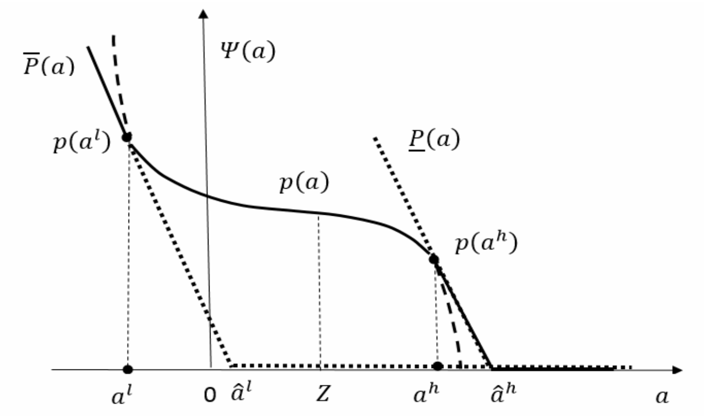

Fig. 3. The equilibrium price function of type C_1

As shown in Fig. 3, the values and the derivatives of functions $p(a)$ and $\overline{P}(a)$ coincide at the switching point a^l. Similarly, the values and the derivatives of $p(a)$ and $\underline{P}(a)$ coincide at the switching point $a^h > a^l$. Thus, the switching points a^l and a^h satisfy the conditions of value matching and smooth pasting. The value-matching conditions require the continuity of $\Psi(a)$ at points a^l and a^h:

(22) $$p\left(a^l\right) = \overline{P}\left(a^l\right), \quad p\left(a^h\right) = \underline{P}\left(a^h\right).$$

At these points, traders are indifferent between trading activity and inactivity. As a converges to a^l or a^h, storage converges to 0 or, correspondingly, to $\overline{s}$ because $s = 0$ at the switching point a^l and $s = \overline{s}$ at the switching point a^h.

The smooth-pasting conditions mean that $\Psi(a)$ is differentiable at the switching points:

(23) $$p'\left(a^l\right) = \overline{P}'\left(a^l\right), \quad p'\left(a^h\right) = \underline{P}'\left(a^h\right)$$

and ensure the absence of arbitrage opportunities in infinitesimal intervals around these points. As one can see in Fig. 3, both switching points are below the kink points of the inverse net demand functions $\overline{P}(a)$ and $\underline{P}(a)$: $a^l < \hat{a}^l$, $a^h < \hat{a}^h$, where $\hat{a}^l = b/\mu$ is the kink point for $\overline{P}(a)$ and $\hat{a}^h = \overline{s} + b/\mu$ is the kink point for $\underline{P}(a)$.

Fig. 4 portrays the synthetic EPF $\Psi(a)$ for an equilibrium price path of type C_2. The EPF under trade $p(a)$ is decreasing and convex-concave. The values and derivatives of functions $p(a)$ and $\overline{P}(a)$ coincide at the lower switching point a^l. The function $p(a)$ intersects $\underline{P}(a)$ at the upper switching point a^h, which is the kink point of $\underline{P}(a)$, $a^h = \hat{a}^h$.

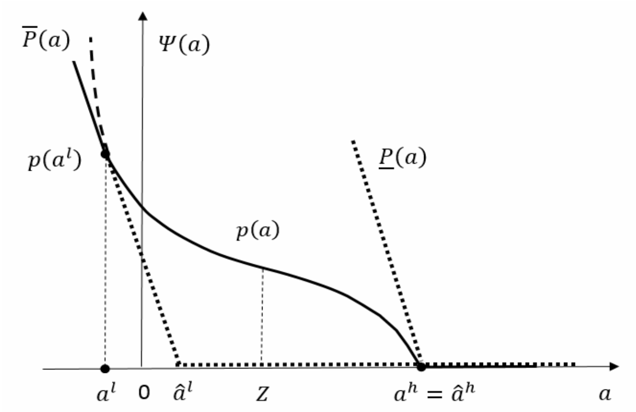

Fig. 4. The equilibrium price function of type C_2

For the EPF of type C_2, the value-matching conditions (22) are fulfilled for both switching points a^l and a^h. However, the smooth-pasting condition is fulfilled only at the lower switching point:

(24) $$p'\left(a^l\right) = \overline{P}'\left(a^l\right),$$

whereas $p'\left(a^h\right)$ varies between $-\mu/\delta$ and 0, as one can see in Fig. 2.

Thus, the structure of the EPFs demonstrated in Fig. 3, 4 is defined by the boundary conditions (22), (23) for type C_1 or (22), (24) for type C_2. The empty storage takes place at higher prices that are above the upper boundary $p\left(a^l\right)$, and $\Psi(a) = \overline{P}(a)$ for $a \le a^l$. The full storage occurs at lower prices that are below the lower boundary $p\left(a^h\right)$, and $\Psi(a) = \underline{P}(a)$ for $a \ge a^h$. Under trade, $\Psi(a) = p(a)$ for $a^l < a < a^h$. The dashed curves in Fig. 3, 4 depict the off-equilibrium continuations of the price function $p(a)$ in the zones of empty storage and full storage. For these continuations, the zero-profit condition $Edp(a) = rp(a)dt$ is fulfilled.

As a result, conditions of value matching and smooth pasting hold for both switching points for type C_1, but for type C_2 the smooth-pasting condition is fulfilled only for a^l, the lower switching point. It does not hold for the upper switching point a^h, but arbitrage is impossible at this point, which coincides with the kink point of $\underline{P}(a)$, $a^h = \hat{a}^h$ (Fig. 4). At this point, traders have no place to store the commodity supplied at zero price. Otherwise, they would have had an opportunity to resell it at a positive price and obtain profits with infinite rate of return.

4.4. Existence and uniqueness of equilibrium

As on can see in Fig. 2, there exists a one-to-one correspondence between the size of storage capacity and the equilibrium price path satisfying the boundary conditions. For zero capacity, the off-equilibrium price path satisfying the no-arbitrage condition (19) is tangent to the horizontal line of inverse net demand P. This path is drawn with the dotted curved arrow C_0. The point of tangency Π in Fig. 2 coincides with the point of intersection of line P and curve Z, the locus of zero second-order derivatives. It is shown in Appendix A that $\Pi = \left(\overline{P}\left(\tilde{a}^l\right),\ -\mu/\delta\right)$, where

$$\tilde{a}^l = \frac{r\left(b/\mu\right)}{r+\mu} = \frac{r\hat{a}^l}{r+\mu}$$

is the solution of equation: $Z\left(\overline{P}(a)\right) = -\mu/\delta$, and the no-arbitrage condition (19) is fulfilled in the absence of speculative trade (for $\overline{s} = 0$) only at price $\overline{P}\left(\tilde{a}^l\right)$ corresponding to the point of tangency.

For any storage capacity $\overline{s}$ above zero, there exists a unique price path satisfying system (20)–(21) and intersecting line P at two points a^l and a^h such that

$$a^l < \tilde{a}^l < a^h, \qquad (25)$$

as shown in Fig. 2. These two points define the boundary conditions and the unique synthetic EPF $\Psi(a)$. If storage capacity is sufficiently small, the equilibrium price path corresponding to $\overline{s}$ is of type C_1. With an increase of this capacity, the trading zone $\left(a^l, a^h\right)$ widens, and the equilibrium price path transforms to type C_2. For storage capacity tending to infinity, the price paths of type C_2 converge to the saddle path S.

4.5. The no-trade conditions

The condition of no-trade under empty storage in (14) is fulfilled as inequality: $Ed\overline{P}(a) < r\overline{P}(a)dt$ (see Appendix B). In the empty storage zone, $a \le a^l$, the expected growth rate is lower for the inverse net demand $\overline{P}(a)$ than for an off-equilibrium continuation of the EPF under trade $p\left(a'\right)$:

(26) $$\frac{Ed\overline{P}(a)}{\overline{P}(a)} < \frac{Edp(a')}{p(a')} = rdt,$$

where $a' = s + x/\mu < a^l$ and $s < 0$.

Similarly, the no-trade condition in the full storage zone, $Ed\underline{P}(a) > r\underline{P}dt$, holds for $a^h < a < \hat{a}^h$ (see Fig. 3), and the expected growth rate is higher for $\underline{P}(a) > 0$ than for an off-equilibrium continuation $p(a')$:

(27) $$\frac{Ed\underline{P}(a)}{\underline{P}(a)} > \frac{Edp(a')}{p(a')} = rdt,$$

where $a^h < a' = s + x/\mu < \hat{a}^h$ and $s > \overline{s}$. Condition (27) is relevant only to the EPFs of type C_1, because $a^h = \hat{a}^h$ for type C_2.

Thus, the expected growth rate of price under empty or full storage can be compared with the growth rate for the virtual price paths, for which it equals the interest rate. In the first case, the expected growth rate for $\overline{P}(a)$ is below this rate for an off-equilibrium continuation of the EPF (drawn on the top of Fig. 3 and 4 with the dashed curves). For this continuation, a virtual accumulation of negative "storage" (or "borrowing" commodity from the future) ensures the no-arbitrage equality in (26) and implies a decrease of the long-term availability that contributes to the price growth. In the second case, the expected growth rate for $\underline{P}(a)$ is above this rate for an off-equilibrium continuation of the EPF of type C_1 (drawn at the bottom of Fig. 3 with the dashed curve). For this continuation, the virtual accumulation of storage above capacity ensures the fulfillment of no-arbitrage equality in (27) and leads to a drop of the virtual price to zero.

There is no trade for the EPFs of both types under full storage and zero price, that is, for $a > \hat{a}^h$.

5. Solutions for a Low-elastic Net Demand

One can assume that parameter δ of net demand function $y(p) = b - \delta p$ is small to solve analytically the system of non-linear differential equations (20)–(21). First, consider a solution for the second-order linear homogenous equation, which is the case for $\delta = 0$:

(28) $$0.5(\sigma/\mu)^2 p''(a) - bp'(a) - rp(a) = 0.$$

Substituting a partial solution $Ae^{\lambda a}$ into (28), where A and λ are unknown parameters, we obtain the quadratic characteristic equation on λ:

$$0.5(\sigma/\mu)^2 \lambda^2 - b\lambda - r = 0.$$

The two roots of this equation are

$$\lambda_{1,2} = \frac{b \pm \sqrt{b^2 + 2(\sigma/\mu)^2 r}}{(\sigma/\mu)^2},$$

so that $\lambda_1 > 0$, $\lambda_2 < 0$. The general solution of equation (28) is a linear combination of partial solutions:

$$p^{(0)}(a) = A_1 e^{\lambda_1 a} + A_2 e^{\lambda_2 a}, \tag{29}$$

where A_1, A_2 are unknowns. One can interpret $p^{(0)}(a)$ as an EPF under inelastic net demand when $\delta = 0$.

Now consider an approximate solution for the non-linear equation (21) rewritten as:

$$0.5(\sigma/\mu)^2 p''(a) - bp'(a) - rp(a) = -\delta p'(a) p(a). \tag{30}$$

Proposition. The first-order approximation for the solution of equation (30) for a small parameter δ is given by:

$$p(a) = p^{(0)}(a) + \delta p^{(1)}(a), \tag{31}$$

where

$$p^{(1)}(a) = \beta_1 \left(A_1 e^{\lambda_1 a}\right)^2 + \beta_{12} A_1 A_2 e^{(\lambda_1 + \lambda_2) a} + \beta_2 \left(A_2 e^{\lambda_2 a}\right)^2, \tag{32}$$

$$\beta_1 = -2\left(b + 3\sqrt{b^2 + 2(\sigma/\mu)^2 r}\right)^{-1}, \quad \beta_2 = 2\left(3\sqrt{b^2 + 2(\sigma/\mu)^2 r} - b\right)^{-1},$$

$$\beta_{12} = -\frac{2b}{r(\sigma/\mu)^2}.$$

Proof: in Appendix C.

The equilibrium price function (31) is the sum of the EPF under inelastic net demand $p^{(0)}(a)$ and the term $\delta p^{(1)}(a)$ related to the market reaction on the price change, which is taken into account by commodity traders. The latter term, as expressed by formula (32), is the quadratic form of the partial solutions of the linear homogenous equation (28).

For the obtained approximate solution given by formulas (31), (29), (32) one have to find two unknown parameters A_1, A_2 and two unknown switching points a^l, a^h. The restrictions imposed on parameters A_1, A_2 are that the sought-for function $p(a)$ must be decreasing and convex-concave in the trading zone $\left(a^l, a^h\right)$.

5.1. Equations implied by the boundary conditions

Let us introduce the new variables:

$$y_{1l} = \delta A_1 e^{\lambda_1 a^l}, \quad y_{2l} = \delta A_2 e^{\lambda_2 a^l}, \tag{33}$$

$$y_{1h} = \delta A_1 e^{\lambda_1 a^h}, \quad y_{2h} = \delta A_2 e^{\lambda_2 a^h}. \tag{34}$$

For an equilibrium price function of type C_1, conditions of value-matching and smooth-pasting (22), (23) can be expressed for these four variables and the switching points a^l, a^h as the six-dimensional system of non-linear equations:

$$y_{1l} + y_{2l} + \beta_1 y_{1l}^2 + \beta_{12} y_{1l} y_{2l} + \beta_2 y_{2l}^2 = b - \mu a^l, \tag{35}$$

$$\lambda_1 y_{1l} + \lambda_2 y_{2l} + 2\lambda_1 \beta_1 y_{1l}^2 + (\lambda_1 + \lambda_2)\beta_{12} y_{1l} y_{2l} + 2\lambda_2 \beta_2 y_{2l}^2 = -\mu, \tag{36}$$

$$y_{1h} + y_{2h} + \beta_1 y_{1h}^2 + \beta_{12} y_{1h} y_{2h} + \beta_2 y_{2h}^2 = b + \mu\left(\overline{s} - a^h\right), \tag{37}$$

$$\lambda_1 y_{1h} + \lambda_2 y_{2h} + 2\lambda_1 \beta_1 y_{1h}^2 + (\lambda_1 + \lambda_2)\beta_{12} y_{1h} y_{2h} + 2\lambda_2 \beta_2 y_{2h}^2 = -\mu, \tag{38}$$

$$\ln y_{1h} - \ln_{1l} = \lambda_1\left(a^h - a^l\right), \tag{39}$$

$$\ln y_{2h} - \ln y_{2l} = \lambda_2\left(a^h - a^l\right). \tag{40}$$

Equations (35)–(38) follow directly from the value-matching and smooth-pasting conditions (22), (23) applied for the EPF (31)–(32), and equations (39), (40) follow from (33), (34).

For an equilibrium price function of type C_2, the smooth-pasting condition does not hold for the upper switching point a^h. This is the kink point for the inverse net demand function $\underline{P}(a)$, such that $\underline{P}\left(a^h\right) = 0$, hence:

$$a^h = \overline{s} + b/\mu. \tag{41}$$

The value-matching condition (22) for this point implies $p\left(a^h\right) = 0$. Consequently, the boundary conditions for the EPF of type C_2 are determined by (41) and the system of five non-linear equations for variables a^l, y_{il}, y_{ih}, $i = 1, 2$:

$$y_{1l} + y_{2l} + \beta_1 y_{1l}^2 + \beta_{12} y_{1l} y_{2l} + \beta_2 y_{2l}^2 = b - \mu a^l, \tag{42}$$

$$\lambda_1 y_{1l} + \lambda_2 y_{2l} + 2\lambda_1 \beta_1 y_{1l}^2 + (\lambda_1 + \lambda_2)\beta_{12} y_{1l} y_{2l} + 2\lambda_2 \beta_2 y_{2l}^2 = -\mu, \tag{43}$$

$$y_{1h} + y_{2h} + \beta_1 y_{1h}^2 + \beta_{12} y_{1h} y_{2h} + \beta_2 y_{2h}^2 = 0, \tag{44}$$

and also two equations identical to (39), (40).

5.2. The limit case: the saddle-path solution

If the storage capacity $\overline{s}$ tends to infinity, then the upper boundary a^h for paths of type C_2 also tends to infinity due to (41). The boundary conditions are

$$p\left(a^l\right)=\overline{P}\left(a^l\right),\quad p'\left(a^l\right)=\overline{P}'\left(a^l\right) \tag{45}$$

and

$$\lim_{a^h\to\infty} p\left(a^h\right)=0,\qquad \lim_{a^h\to\infty} p'\left(a^h\right)=0. \tag{46}$$

As one can see from the phase plane in Fig. 2, the limit boundary conditions (46) are fulfilled for the saddle path S.

The paths of type C_2 converge to the saddle path as $\overline{s}$ tends to infinity, and in the limit the capacity constraint $s\le\overline{s}$ does not affect the price. A solution for the saddle path is the special case of the EPF (31)–(32), when $A_1=0$ and the terms with positive exponent $A_1e^{\lambda_1 a}$ vanish:

$$p(a)=A_2e^{\lambda_2 a}+\delta\beta_2\left(A_2e^{\lambda_2 a}\right)^2. \tag{31′}$$

For the term with negative exponent $A_2e^{\lambda_2 a}$ in (31′), we have it that $A_2>0$. From (33), condition $A_1=0$ implies that $y_{1l}=0$. The boundary conditions (45) are represented for the saddle-path solution as:

$$2\beta_2 y_{2l}^2+y_{2l}+\mu/\lambda_2=0, \tag{47}$$

$$\mu a^l=b-y_{2l}-\beta_2 y_{2l}^2. \tag{48}$$

These are equations (43), (42) rewritten for $y_{1l}=0$, and the unknown variables are y_{2l}, a^l. Parameter $A_2>0$ is found from (33) as $A_2=y_{2l}/\left(\delta e^{\lambda_2 a^l}\right)$. The system (47), (48) is solved explicitly: since the intercept in the square equation (47) is negative (because $\lambda_2<0$), the positive real root of this equation y_{2l} is inserted into (48) to find the boundary a^l.

6. Numerical Simulations

Now we can calculate numerically the equilibrium price functions for arbitrarily selected bundles of exogenous model parameters: r, b, δ, σ, μ, $\overline{s}$. For each bundle, we find, by solving the system of equations (35)–(40) or (39)–(44), the unknown parameters A_1, A_2 and the variables a^l, a^h that define the EPF under trade $p(a)$ and the synthetic equilibrium price function $\Psi(a)$. These functions are calculated by the formulas (31)–(32), (29) for the two types of equilibrium solution: C_1 and C_2. The results of simulations are presented graphically below.

6.1. Equilibrium price functions of type C_1

Consider a numerical example for the following values of the model parameters: $r = 0.015$, $b = 1$, $\delta = 0.05$, $\sigma = 6$, $\mu = 2$, $\overline{s} = 10$. The solution for the unknown variables a^l, a^h and parameters A_1, A_2 is found from the system (35)–(40), which is solved for six variables: a^l, a^h, y_{1l}, y_{1h}, y_{2l}, y_{2h}. Parameters A_1, A_2 are calculated from equalities (33), (34) that must hold as identities for any solution of this system.

The synthetic EPF $\Psi(a)$ is depicted in Fig. 5. It consists of three pieces pasted together: the equilibrium price functions under trade $p(a)$ and the inverse net demand functions under empty storage $\overline{P}(a)$ and full storage $\underline{P}(a)$. One can see from this figure that the value-matching and smooth-pasting conditions (22), (23) are satisfied at the switching points $a^l = -32.58$ and $a^h = 2.03$. The dashed curves show the off-equilibrium continuations of $p(a)$ in the zones of empty storage, $a \le a^l$, and full storage, $a \ge a^h$.

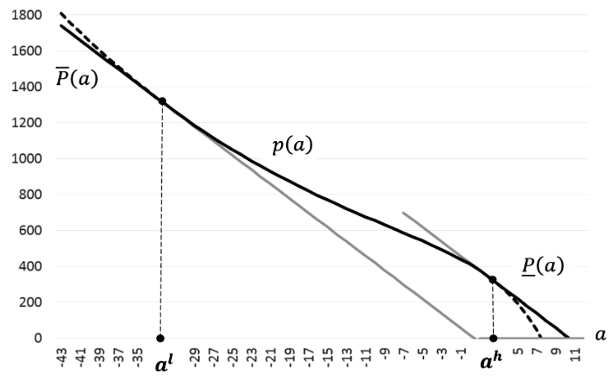

Fig. 5. The synthetic equilibrium price function $\Psi(a)$

For the same numerical example we consider the interest rate change from $r = 0.015$ to $r = 0.02$, while leaving other parameters the same. Fig. 6 demonstrates the effects of the interest rate increase on the EPF. The inverse net demand functions $\overline{P}(a)$ and $\underline{P}(a)$ drawn with grey lines do not alter, whereas the trading zone $\left(a^l, a^h\right)$ changes notably: it shifts to the right and narrows.

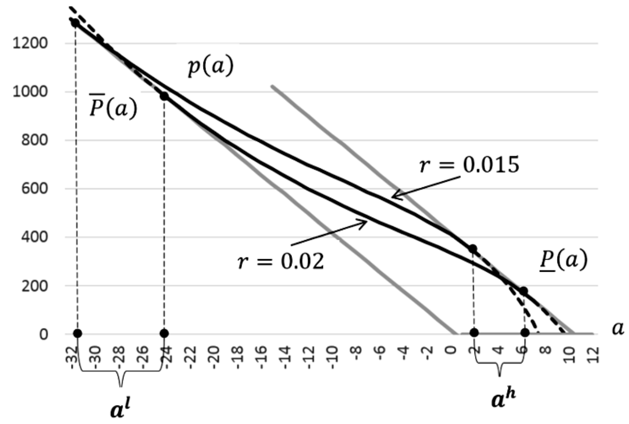

Fig. 6. The EPFs of type C_1 for different interest rates

Fig. 6 also demonstrates that a lower interest rate results in a higher level of equilibrium prices. A lower interest rate is more favourable for investment in commodity for the following reason. From the no-trade conditions in (14), buying the commodity under empty storage is unprofitable: $Ed\overline{P}(a) < r\overline{P}(a)dt$, and the zone $a \le a^l$ satisfying this condition narrows for a lower interest rate. In contrast, a higher interest rate brings about a lower price level because it is more favourable for sales of commodity by traders. The no-trade condition under full storage, $Ed\underline{P}(a) > r\underline{P}(a)dt$, makes sales of commodity unprofitable. The zone $a \ge a^h$ satisfying this condition (for positive prices) narrows under a higher interest rate.

6.2. Equilibrium price functions of type C_2

Numerical simulations for the price function of type C_1 show that for a sufficiently large storage capacity $\overline{s}$ the price at the upper boundary of trading zone a^h becomes negative, $p\left(a^h\right) < 0$. In such case, one have to search for a solution of type C_2. It satisfies the system of equations (39)–(44) with the upper boundary a^h calculated as (41) and without the condition of smooth pasting (38) for this boundary.

Consider a numerical example with parameter values: $r = 0.03$, $b = 1$, $\delta = 0.05$, $\sigma = 7$, $\mu = 0.4$. Fig. 7 demonstrates the EPFs of type C_2 for different values of storage capacity: $\overline{s} = 20, 40, 80$. All the lower boundaries a^l locate near the value $a = -54$, while the upper boundaries a^h, for which $p\left(a^h\right)=0$, spread along the horizontal axis. For any EPF of this type, the storage capacity is filled when the commodity price drops to zero. As one can see from Fig. 7, the point of switch to full storage a^h increases with $\overline{s}$ implying a higher level of prices in the trading zone $\left(a^l, a^h\right)$. The ability of traders to accumulate greater volumes of storage is captured in price expectations that exert an upward pressure on prices.

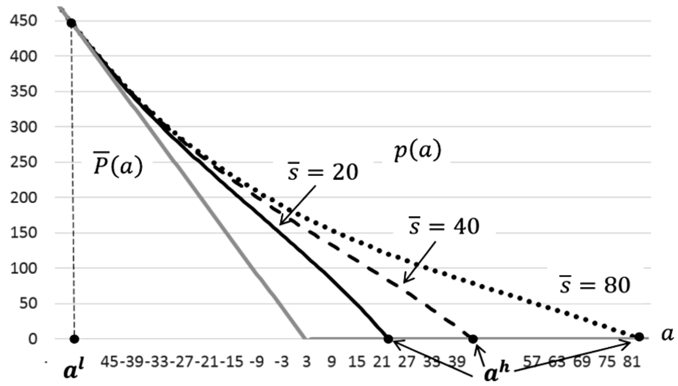

Fig. 7. The EPFs of type C_2 for various storage capacities

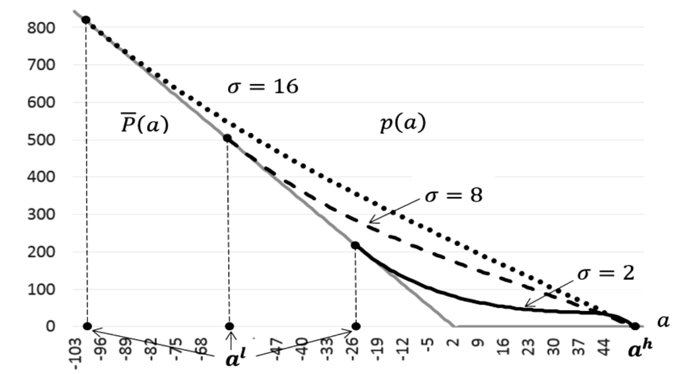

Fig. 8. The EPFs of type C_2 for various standard deviations

Fig. 8 shows the EPFs for different values of the standard deviation of net supply process: $\sigma = 2, 8, 16$, the storage capacity $\overline{s} = 50$, and other parameters the same as in the previous example. One can see from the figure that the upper switching point a^h is invariant for all functions $p(a)$, because a^h does not depend on σ, due to condition (41). The lower switching point a^l shifts to the left and the trading zone widens with an increase of standard deviation.

The threshold price $p\left(a^l\right)=\overline{P}\left(a^l\right)$ increases with σ, and this causes an increase of the price level for all a in the trading zone. The reason is that under a higher volatility of net supply disturbances, a larger volume of storage is required to ensure the fulfilment of no-arbitrage condition (19) implying a higher speculative demand for commodity.

6.3. Equilibrium price functions for the limit case $\overline{s}=\infty$

Finally, consider the case of unlimited storage capacity. The EPF under trade $p(a)$ for this case is given by formula (31′) and convex for the trading zone $a \ge a^l$. The solution for the synthetic price function $\Psi(a)$ is defined by equations (47), (48) for variables a^l and y_{2l}.

The EPFs depicted in Fig. 9 are calculated for the numerical example: $r=0.03$, $b=1$, $\delta=0.05$, $\mu=0.3$ and $\sigma=0.5, 5, 10$. As in the previous example, the equilibrium price level is higher for a larger variance of net supply process.

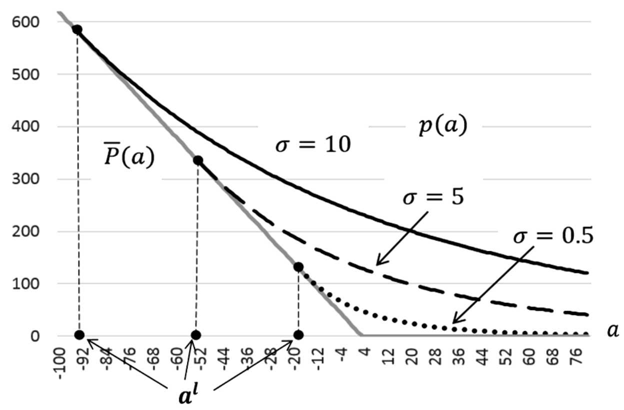

Fig. 9. The EPFs for various standard deviations, $\overline{s}=\infty$

This figure and the previous ones illustrate the real-option properties of commodity prices. The real-option valuation is a consequence of the dual nature of commodity as a consumption good and a storable asset. Traders at any instant have an option to store a unit of commodity in order to sell it in the future at the best price [Karimi et al., 2021]. The value of storage option equals the difference between equilibrium price and inverse net demand: $p(a)-\overline{P}(a)$. As one can see from Fig. 9, this option value is increasing with the volatility of net supply disturbances.

7. Conclusions

We have considered the competitive storage model in continuous time with the stochastic process of net supply and bounded storage capacity. The relevance of this approach is justified by the technical convenience it brings, as well as by its suitability to the mineral markets in which the time series data is available for daily frequency. The introduction of an upper-bound constraint on storage allows the model to reflect the phenomena of supply gluts and negative price spikes that have happened in commodity markets.

The focus of this article was on the formal analysis of the theoretical model of competitive storage. The stochastic variable of net supply in the model relates to disturbances on demand and supply sides. We considered serially correlated disturbances and characterized the equilibrium price function in this framework. The key point of the model solution is the choice of the state variable that determines the equilibrium price. For the continuous-time model, the state variable is the long-term availability defined as the sum of storage and the expected cumulative net supply disturbances. The latter term reflects the expectations of traders with regard to future shifts on demand and supply sides affecting the current price.

There is no contradiction between our model and the discrete-time storage models pointed out in the introduction (e.g. [Deaton, Laroque, 1992]), where the state variable is current availability. The latter coincides, up to a constant term, with the long-term availability as defined here, under serially uncorrelated supply processes assumed in those models.

For the stationary rational expectations equilibrium of our model, the exogenous stochastic process of net supply disturbances was replaced with the endogenous process for the long-term availability. We derived the non-linear differential equation for the equilibrium price function of the state variable. The free-boundary conditions determine the trading zone and the synthetic EPF as a partial solution of this equation. The approximation of this solution for a low-elastic net demand makes it possible to conduct simple numerical calculations of the EPFs for various bundles of the model parameters.

These calculations demonstrate the properties of the equilibrium price functions revealed by the qualitative model analysis. It was shown for numerical examples that there are two types of these functions, which are convex-concave and decreasing, and that the trading zone widens and the price level increases with a decrease of the interest rate and an increase of the storage capacity or the variance of net supply disturbances.

The model was examined for the case of a piecewise-linear net demand. It can be extended to a more general net demand function $y(p)$, for which the inverse net demand $P(x)$ is given implicitly. In such a case, the boundary conditions (35)–(44) are changed. In particular, the value-matching conditions (35), (37) or (42), (44) cannot be written explicitly because their right-hand sides are given by the implicit functions of inverse net demand $\overline{P}\left(a^l\right)$ or $\underline{P}\left(a^h\right)$. Therefore, the value-matching conditions in general case should be determined through an iterative scheme, whereas the smooth-pasting conditions can be obtained explicitly by implicit differentiation of inverse net demand functions.

The model parameters were selected arbitrarily for numerical calculations. It is possible to estimate parameters of the net demand function and the stochastic process of net supply for the real-world commodity markets by using detrended price series and commercial storage data. With the econometric estimates of parameters, one can calculate the net supply disturbances and the long-term availability series and then evaluate the equilibrium price functions for the functional forms derived in this article.

Appendix A.

The point of tangency Π

Consider the equation: $Z(p) = -\mu/\delta$, where $Z(p) = -rp/\left(b - \delta p\right)$ corresponds to curve Z in Fig. 2. The solution is given by

$$\tilde{p} = \frac{\mu\left(b/\delta\right)}{r+\mu}.$$

The solution for equation $\tilde{p} = \overline{P}(a)$, where $\overline{P}(a) = \left(b - \mu a\right)/\delta$, is

$$\tilde{a}^l = \frac{r\left(b/\mu\right)}{r+\mu} = \frac{r\hat{a}^l}{r+\mu},$$

hence, $\tilde{a}^l$ is the solution of equation: $Z\left(\overline{P}(a)\right) = -\mu/\delta$. For $\overline{s} = 0$, the no-arbitrage condition (19) is fulfilled only at point $\tilde{a}^l$, which is the point of tangency. Indeed, from (17), $Eda = -xdt = -\mu adt$, and the difference $Ed\overline{P}(a) - r\overline{P}(a)dt$ is linear and increasing in a:

$$\begin{aligned} &Ed\overline{P}(a) - r\overline{P}(a)dt = \overline{P}'(a)Eda - r\left(b - \mu a\right)dt/\delta = \\ &= \left(\mu/\delta\right)\mu adt + \left(\mu/\delta\right)radt - \left(rb/\delta\right)dt = \left(\left(\mu/\delta\right)\left(\mu + r\right)a - \left(rb/\delta\right)\right)dt, \end{aligned}$$

since $\overline{P}''(a) \equiv 0$, $\overline{P}'(a) = -\mu/\delta$. This difference equals zero only at point $\tilde{a}^l$.

Appendix B.

Inequalities (26), (27)

It follows from the proof in Appendix A that inequality (26), $Ed\overline{P}(a) < r\overline{P}(a)dt$, is fulfilled for $a \le a^l$ because $a^l < \tilde{a}^l$ from (25) and the difference $Ed\overline{P}(a) - r\overline{P}(a)dt$ is increasing in a. Consider inequality (27) and rearrange the same difference for $\underline{P}(a) > 0$:

$$E d\underline{P}(a) - r\underline{P}(a)dt = \underline{P}'(a)Eda - r\left(b + \mu\left(\bar{s} - a\right)\right)dt/\delta =$$
$$= \left(\mu/\delta\right)\mu a dt + \left(\mu/\delta\right) r a dt - \left(r\left(b + \mu\bar{s}\right)/\delta\right)dt =$$
$$= \left(\left(\mu/\delta\right)\left(\mu + r\right)a - r\left(b + \mu\bar{s}\right)/\delta\right)dt,$$

since $\underline{P}''(a) \equiv 0$, $\underline{P}'(a) = -\mu/\delta$, $Eda = -\mu a dt$. Consequently, $Ed\underline{P}(a) > r\underline{P}(a)dt$ if

$$a > \tilde{a}^h = \frac{r\left(b + \mu\bar{s}\right)}{\left(r + \mu\right)\mu} = \frac{r\hat{a}^h}{r + \mu}.$$

It can be shown, similarly to the proof in Appendix A, that $\tilde{a}^h$ is the solution of equation $Z\left(\underline{P}(a)\right) = -\mu/\delta$. This implies $\underline{P}\left(\tilde{a}^h\right) = \overline{P}\left(\tilde{a}^l\right) = \tilde{p}$ and the chain of inequalities extending (25): $a^l < \tilde{a}^l < \tilde{a}^h < a^h$. Since the difference $Ed\underline{P}(a) - r\underline{P}(a)dt$ is increasing in a, we have it that $Ed\underline{P}(a) > r\underline{P}(a)dt$ for $a \ge a^h > \tilde{a}^h$ $\left(a < \hat{a}^h\right)$.

Appendix C.

Proposition

One can represent a solution of (30) as the function series expansion for degrees of δ:

(C.1) $$p(a) = p^{(0)}(a) + \delta p^{(1)}(a) + \delta^2 p^{(2)}(a) + \ldots .$$

Inserting this into (30) implies the relationship:

$$0.5\left(\sigma/\mu\right)^2\left(p^{(0)\prime\prime} + \delta p^{(1)\prime\prime} + \delta^2 p^{(2)\prime\prime} + \ldots\right) - b\left(p^{(0)\prime} + \delta p^{(1)\prime} + \delta^2 p^{(2)\prime} + \ldots\right) -$$
$$-r\left(p^{(0)} + \delta p^{(1)} + \delta^2 p^{(2)} + \ldots\right) = -\delta\left(p^{(0)\prime} + \delta p^{(1)\prime} + \delta^2 p^{(2)\prime} + \ldots\right)\left(p^{(0)} + \delta p^{(1)} + \delta^2 p^{(2)} + \ldots\right).$$

Collecting terms with the same degrees of δ yields the system of interconnected equations:

(C.2) $$\delta^0: \quad 0.5\left(\sigma/\mu\right)^2 p^{(0)\prime\prime} - bp^{(0)\prime} - rp^{(0)} = 0$$

(C.3) $$\delta^1: \quad 0.5\left(\sigma/\mu\right)^2 p^{(1)\prime\prime} - bp^{(1)\prime} - rp^{(1)} = -p^{(0)\prime}p^{(0)}$$

$$\delta^2: \quad 0.5\left(\sigma/\mu\right)^2 p^{(2)\prime\prime} - bp^{(2)\prime} - rp^{(2)} = -p^{(0)\prime}p^{(1)} - p^{(1)\prime}p^{(0)}$$

…….

Equations corresponding to the degrees of δ higher than one are ruled out. Equation (C.2) is the linear homogenous equation identical to (28) with the general solution $p^{(0)}(a) = A_1 e^{\lambda_1 a} + A_2 e^{\lambda_2 a}$. The derivative is $p^{(0)\prime}(a) = \lambda_1 A_1 e^{\lambda_1 a} + \lambda_2 A_2 e^{\lambda_2 a}$. Inserting this into the right-hand side of (C.3) yields the non-homogenous linear equation:

(C.4)
$$0.5(\sigma/\mu)^2 p^{(1)\prime\prime}(a) - bp^{(1)\prime}(a) - rp^{(1)}(a) = \\ = \lambda_1 A_1^2 e^{2\lambda_1 a} + (\lambda_1 + \lambda_2) A_1 A_2 e^{(\lambda_1+\lambda_2)a} + \lambda_2 A_2^2 e^{2\lambda_2 a}.$$

We will find a partial solution of this equation as

(C.5)
$$p^{(1)}(a) = B_1 e^{2\lambda_1 a} + B_{12} e^{(\lambda_1+\lambda_2)a} + B_2 e^{2\lambda_2 a}.$$

where B_1, B_{12}, B_2 are unknown parameters. Substituting this into (C.4) and collecting terms with identical exponents yields equations on B_i, $i = 1,2$, and B_{12}, respectively:

(C.6)
$$2(\sigma/\mu)^2 \lambda_i^2 B_i e^{2\lambda_i a} - 2b\lambda_i B_i e^{2\lambda_i a} - rB_i e^{2\lambda_i a} = -\lambda_i A_i^2 e^{2\lambda_i a}, \quad i = 1,2$$

(C.7)
$$0.5(\sigma/\mu)^2 (\lambda_1 + \lambda_2)^2 B_{12} e^{(\lambda_1+\lambda_2)a} - b(\lambda_1 + \lambda_2) B_{12} e^{(\lambda_1+\lambda_2)a} - rB_{12} e^{(\lambda_1+\lambda_2)a} = \\ = (\lambda_1 + \lambda_2) A_1 A_2 e^{(\lambda_1+\lambda_2)a}.$$

Rearrange (C.6) to solve for B_i:

$$B_i \left(2(\sigma/\mu)^2 \lambda_i^2 - 2b\lambda_i - r\right) = -\lambda_i A_i^2.$$

The left-hand side of this equation equals $B_i \left((3/2)(\sigma/\mu)^2 \lambda_i^2 - b\lambda_i\right)$ because λ_i satisfies the characteristic equation: $0.5(\sigma/\mu)^2 \lambda_i^2 - b\lambda_i - r = 0$. Consequently,

$$B_i = \frac{-\lambda_i A_i^2}{(3/2)(\sigma/\mu)^2 \lambda_i^2 - b\lambda_i} = \frac{-2A_i^2}{3(\sigma/\mu)^2 \lambda_i - 2b}$$

for $i = 1,2$. This implies:

$$B_1 = \frac{-2A_1^2}{b + 3\sqrt{b^2 + 2(\sigma/\mu)^2 r}}, \quad B_2 = \frac{2A_2^2}{3\sqrt{b^2 + 2(\sigma/\mu)^2 r} - b}$$

because $\lambda_{1,2} = \dfrac{b \pm \sqrt{b^2 + 2(\sigma/\mu)^2 r}}{(\sigma/\mu)^2}$.

Rearrange (C.7) to solve for B_{12}:

$$B_{12}\left(0.5(\sigma/\mu)^2(\lambda_1+\lambda_2)^2 - b(\lambda_1+\lambda_2) - r\right) = (\lambda_1+\lambda_2)A_1A_2,$$

$$B_{12}\left(\frac{2b^2}{(\sigma/\mu)^2} - \frac{2b^2}{(\sigma/\mu)^2} - r\right) = \frac{2b}{(\sigma/\mu)^2}A_1A_2,$$

because $\lambda_1 + \lambda_2 = 2b/(\sigma/\mu)^2$. Hence,

$$B_{12} = -\frac{2bA_1A_2}{r(\sigma/\mu)^2}.$$

From (C.1), (29), (C.5), the solution of equation (30) is

$$p(a) \approx p^{(0)}(a) + \delta p^{(1)}(a) = A_1e^{\lambda_1 a} + A_2e^{\lambda_2 a} + \delta\left(B_1e^{2\lambda_1 a} + B_{12}e^{(\lambda_1+\lambda_2)a} + B_2e^{2\lambda_2 a}\right) =$$

$$= A_1e^{\lambda_1 a} + A_2e^{\lambda_2 a} + \delta\left(\frac{-2A_1^2e^{2\lambda_1 a}}{b + 3\sqrt{b^2 + 2(\sigma/\mu)^2 r}} - \frac{2bA_1A_2e^{(\lambda_1+\lambda_2)a}}{r(\sigma/\mu)^2} + \frac{2A_2^2e^{2\lambda_2 a}}{3\sqrt{b^2 + 2(\sigma/\mu)^2 r} - b}\right) =$$

$$= A_1e^{\lambda_1 a} + A_2e^{\lambda_2 a} + \delta\left(\beta_1\left(A_1e^{\lambda_1 a}\right)^2 + \beta_{12}A_1A_2e^{(\lambda_1+\lambda_2)a} + \beta_2\left(A_2e^{\lambda_2 a}\right)^2\right).$$

* *
*

References

Cafiero O., Bobenrieth E., Zeldes S. (2015) The Empirical Relevance of the Competitive Storage Model. *Journal of Econometrics*, 162, 1, pp. 44–54.

Chambers M., Bailey R. (1996) A Theory of Commodity Price Fluctuations. *Journal of Political Economy,* 104, 5, pp. 924–957.

Deaton A., Laroque G. (1992) On the Behaviour of Commodity Prices. *The Review of Economic Studies*, 59, 1, pp. 1–23.

Deaton A., Laroque G. (1995) Estimating a Non-linear Rational Expectations Commodity Price Model with Unobservable State Variables. *Journal of Applied Econometrics*, 10, pp. S9–S40.

Deaton A., Laroque G. (1996) Competitive Storage And Commodity Price Dynamics. *Journal of Political Economy*, 104, 5, pp. 896–923.

Dvir E., Rogoff K. (2009) *Three Epochs of Oil*. NBER Working Paper no 14927.

Gibson R., Schwartz E. (1990) Stochastic Convenience Yield and the Pricing of Oil Contingent Claims. *Journal of Finance*, 45, 3, pp. 959–976.

Guerra E., Bobenreith E., Bobenreith J., Cafiero C (2015) Empirical Commodity Storage Model: The Challenge of Matching Data and Theory. *European Review of Agricultural Economics,* 42, 4,) pp. 607–623.

Gustafson R. (1958) Carryover Levels for Grains: A Method for Determining Amounts that Are Optimal under Specified Conditions. *Technical Bulletin*, 178, US Dept. of Agriculture.

Karimi N., Assa H., Salavati E., Adibi H. (2021) *Commodity Price Modelling by Optimal Storage Time*. Available at: https://www.researchgate.net/publication/351081296_Commodity_price_modeling_by_optimal_storage_time

Oglend A., Kleppe T. (2017) On the Behaviour of Commodity Prices When Speculative Storage Is Bounded. *Journal of Economic Dynamics and Control*, 75, pp. 5–69.

Ribeiro D., Hodges S. (2004) *A Two-Factor Model for Commodity Prices and Futures Valuation: SSRN*. Available at: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=498802.

Schwartz E. (1997) The Stochastic Behaviour of Commodity Prices: Implications for Valuation and Hedging. *Journal of Finance*, 52, 3, pp. 923–973.

Vavilov A., Trofimov G. (2021) *Natural Resource Pricing and Rents, chapter 8: Commodity trade in continuous time, long-term availability and storage capacity*. Springer, Cham., pp. 197–221.

Wright B. (2001) Storage and Price Stabilization. *Handbook of Agricultural Economics*, 1, pp. 817–861.